DGFEM for Hamilton–Jacobi–Bellman equations with Cordes coefficients

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17th IMA Leslie Fox Prize Meeting

joint work with

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Overview

Talk outline

- *Introduction:* Stochastic optimal control and Hamilton–Jacobi–Bellman (HJB) equations.
- *PDE Theory:* Analysis of HJB equations with Cordes coefficients.
- *Numerical methods:* High-order discontinuous Galerkin methods for HJB equations with Cordes coefficients.

Overview

Talk outline

- *Introduction:* Stochastic optimal control and Hamilton–Jacobi–Bellman (HJB) equations.
  - How HJB equations arise from stochastic optimal control problems.
  - Some example applications.
  - Examples of a broad class of fully nonlinear equations.

- *PDE Theory:* Analysis of HJB equations with Cordes coefficients.
- *Numerical methods:* High-order discontinuous Galerkin methods for HJB equations with Cordes coefficients.

What is a stochastic optimal control problem?

Find a control function $\alpha(\cdot)\colon t \mapsto \alpha_t$ that minimises

$$J(x,\alpha(\cdot)) = \mathbb{E}\left[\int_0^{\tau_{\mathrm{exit}}} f(X_t,\alpha_t)\exp\left(-\int_0^t c(X_s,\alpha_s)\,\mathrm{d}s\right)\mathrm{d}t\right]$$

subject to the stochastic differential equation

$$\mathrm{d}X_t = b(X_t,\alpha_t)\,\mathrm{d}t + \sigma(X_t,\alpha_t)\,\mathrm{d}B_t, \quad X_0 = x.$$

Notation: $\alpha \in \Lambda$ the set of controls, $x \in \Omega$ a domain in $\mathbb{R}^d$,
$\mathrm{d}B_t$ a m-dimensional Brownian motion, τ_{exit} is the time of first exit of X_t from Ω.

$$\sigma(x,\alpha) \in \mathbb{R}^{d\times m}, \qquad b(x,\alpha) \in \mathbb{R}^d, \qquad c(x,\alpha) \in \mathbb{R}, \qquad f(x,\alpha) \in \mathbb{R}.$$

$\rightarrow$ *an optimisation problem over a function space.*

1. Stochastic optimal control: example applications


Engineering


Energy


Finance

Applications of optimal control theory are covered in a very wide literature. A couple of examples:

I. Karatzas, J. Lehoczky & S. Shreve, SIAM J. Control Optim. 1987:
Optimal portfolio and consumption decisions for a "small investor" on a finite horizon.

P. Parpas & M. Webster, Eur. J. Op. Res. 2013:
A stochastic multiscale model for electricity generation capacity expansion.

1. Stochastic optimal control: dynamic programming principle

How does the HJB equation arise in stochastic control problems?

Bellman's dynamic programming principle

1. Define the *value function* of the optimal control problem.
2. DPP: the value function is the solution of an HJB equation.
3. Solving the HJB equation yields the value function and the optimal controls.

Full details in many references, e.g. [Fleming & Soner, 2006].

Richard Bellman (1920–1984)

1.HJB equation

$$\begin{aligned} \sup_{\alpha\in\Lambda}[L^\alpha u - f^\alpha] &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{HJB}$$

where $L^\alpha u := a^\alpha(x) : D^2 u + b^\alpha(x)\cdot\nabla u - c^\alpha(x)\,u$, with

$$\begin{aligned} a^\alpha(x) &:= \tfrac{1}{2}\sigma(x,\alpha)\,\sigma^\top(x,\alpha) \in \mathbb{R}^{d\times d}, \qquad & b^\alpha(x) &:= b(x,\alpha)\in\mathbb{R}^d, \\ c^\alpha(x) &:= c(x,\alpha)\in\mathbb{R}, & f^\alpha(x) &:= f(x,\alpha)\in\mathbb{R}. \end{aligned}$$

Notation: $a^\alpha(x) : D^2 u = \sum_{i,j=1}^{d} a^\alpha_{ij}(x) u_{x_i x_j}, \quad b^\alpha(x)\cdot\nabla u = \sum_{i=1}^{d} b^\alpha_i(x) u_{x_i}.$

Remark: the dependence of a function on $\alpha\in\Lambda$ is denoted by a superscript: $a\colon (x,\alpha)\to a^\alpha(x)$.

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1. HJB equation: examples

How do HJB equations relate to other partial differential equations?

$$\sup_{\alpha \in \Lambda} [L^\alpha u - f^\alpha] = 0$$

The HJB equation generalises many other equations:

- Linear nondivergence form elliptic equations

$$a : D^2 u + b \cdot \nabla u - cu = f, \quad \text{(assume that } \Lambda \text{ is a singleton set).}$$

- Linear advection–diffusion–reaction equation

$$\operatorname{div}(a \nabla u) + b \cdot \nabla u - cu = f, \quad \text{(assuming } a \in C^1(\overline{\Omega})\text{).}$$

- Hamilton–Jacobi: e.g. eikonal equation

$$\sup_{\alpha \in \mathbb{S}^d} [\alpha \cdot \nabla u - 1] = |\nabla u| - 1 = 0.$$

- Monge–Ampère equation

$$\det D^2 u - f = 0 \iff \inf_{\substack{\alpha \in \mathbb{R}^{d \times d}_{\mathrm{sym},+} \\ \operatorname{Tr} \alpha = 1}} \left[\alpha : D^2 u - d \, (f \, \det \alpha)^{1/d} \right] = 0.$$

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Approach to numerical methods:

- Different numerical methods are commonly used for different special cases with specific structures (e.g. eikonal vs advection-diffusion-reaction).
- Our goal is to develop stable and efficient numerical methods for a subclass of HJB equations with a common structure.
- Previous methods either:
  - had a convergence theory, but relied on discrete maximum principles and restricted to low-order schemes.

    cf works of Motzkin & Wasow, Kuo & Trudinger, Barles & Souganidis, Kocan, Camilli & Falcone, Jakobsen & Debrabant, and many others...

  - did not rely on discrete maximum principles, but had no convergence theory.

    cf review paper [Feng et al., SIAM Rev, 2013].

1.HJB Equation: summary

1. Stochastic optimal control problems lead to HJB equations.

2. HJB equations are a broad class of *fully nonlinear* partial differential equations. They generalise many other PDE.

3. The nonlinearity leads to multiple challenges in designing stable and convergent numerical methods.

Overview

Talk outline

- *Introduction:* Stochastic optimal control and Hamilton–Jacobi–Bellman (HJB) equations.
- *PDE Theory:* Analysis of HJB equations with Cordes coefficients.
  - What are Cordes coefficients, where do they come from?
  - Why are they relevant to HJB equations?
  - What is the PDE analysis of HJB equations with Cordes coefficients (existence, uniqueness and regularity of solutions)
- *Numerical methods:* High-order discontinuous Galerkin methods for HJB equations with Cordes coefficients.

2. PDE theory: Hamilton–Jacobi–Bellman Equation

We will consider the elliptic HJB equation

$$\begin{aligned} \sup_{\alpha\in\Lambda}[L^\alpha u - f^\alpha] &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \qquad \text{(Elliptic HJB)}$$

where $L^\alpha u := a^\alpha(x) : D^2 u + b^\alpha(x)\cdot\nabla u - c^\alpha(x)\, u$.

Assumptions:

- $\Omega \subset \mathbb{R}^d$ is bounded and convex, Λ a compact metric space.
- a, b, c and f are continuous functions in $x \in \overline{\Omega}$, $\alpha \in \Lambda$.
- a^α are symmetric positive definite, uniformly on $\overline{\Omega} \times \Lambda$, and $c^\alpha \geq 0$.
- Cordes coefficients: the coefficient functions a, b, c satisfy *the Cordes condition* (cf next slide)

2. PDE theory: Cordes condition

Cordes condition: assume that there exist $\lambda > 0$ and $\varepsilon \in (0,1]$ s. t.

$$\frac{|a^\alpha|^2 + |b^\alpha|^2/2\lambda + (c^\alpha/\lambda)^2}{(\operatorname{Tr} a^\alpha + c^\alpha/\lambda)^2} \leq \frac{1}{d+\varepsilon} \quad \forall\, \alpha \in \Lambda. \qquad \text{(Cordes)}$$

If $b^\alpha \equiv 0$ and $c^\alpha \equiv 0$, then (Cordes) is modified slightly:

$$\frac{|a^\alpha|^2}{(\operatorname{Tr} a^\alpha)^2} \leq \frac{1}{d-1+\varepsilon} \quad \forall\, \alpha \in \Lambda. \qquad (\text{Cordes}^2)$$

If dimension $d = 2$, (Cordes^2) equivalent to uniform ellipticity $\Longrightarrow$ widely applicable.

$$\textit{Notation:} \quad |a^\alpha|^2 = \sum_{i,j=1}^{d} |a^\alpha_{ij}|^2, \quad \operatorname{Tr} a^\alpha = \sum_{i=1}^{d} a^\alpha_{ii}, \quad |b^\alpha|^2 = \sum_{i=1}^{d} |b^\alpha_i|^2.$$

2. PDE theory: Cordes condition

The Cordes condition: [Cordes, 1956] $\to$ PDE theory of nondivergence form elliptic equations with discontinuous coefficients.

$$a\colon D^2u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad a \in L^\infty(\Omega). \tag{1}$$

If $a \in C(\overline{\Omega})$ and $\partial\Omega \in C^{1,1}$, then existence and uniqueness holds.

[Gilbarg & Trudinger].

If $a \in L^\infty(\Omega)$, $a \notin C(\Omega)$, then uniqueness breaks down:

$$\Delta u + \rho \sum_{i,j=1}^{d} \frac{x_i x_j}{|x|^2} u_{x_i x_j} = 0 \text{ in } B, \qquad \rho = -1 + \frac{d-1}{1-\theta}, \quad 0 < \theta < 1,$$

where B is the unit ball in $\mathbb{R}^d$. For $d > 2(2-\theta) > 2$, the equation has two solutions

$$u_1(x) = 0 \qquad \text{and} \qquad u_2(x) = |x|^\theta - 1$$

in $H^2(B) \cap H^1_0(B)$, whenever $d \geq 3$.

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$$a : D^2 u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad a \in L^\infty(\Omega). \tag{1}$$

Theorem (Cordes, 1956)

If $a \in L^\infty(\Omega)$, Ω is convex, and the Cordes condition holds,

$$\frac{|a|^2}{(\operatorname{Tr} a)^2} \leq \frac{1}{d - 1 + \varepsilon}$$

then for any $f \in L^2(\Omega)$ there exists a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ satisfying (1).

Our motivation:

Recall that HJB generalises nondivergence form elliptic equations.

More specifically, nondivergence form elliptic equations *with discontinuous coefficients* arise as linearisations of HJB equations.

It is therefore natural to study the subclass of HJB equations that satisfy the Cordes condition.

Theorem

Let Ω be a bounded convex open subset of $\mathbb{R}^d$, and let Λ be a compact metric space.

Let the data be continuous on $\overline{\Omega} \times \Lambda$, and satisfy (Cordes) *with uniformly elliptic a^α and $c^\alpha \geq 0$ for all $\alpha \in \Lambda$.*

Then, there exists a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ that solves (HJB) *pointwise a.e. in Ω.*

I. S. & E. Süli, SIAM J. Numer. Anal. 2014:

Discontinuous Galerkin finite element approximation of Hamilton–Jacobi–Bellman equations with Cordes coefficients.

2. PDE theory: proof of well-posedness

Define

$$\gamma^\alpha := \frac{\operatorname{Tr} a^\alpha + c^\alpha/\lambda}{|a^\alpha|^2 + |b^\alpha|^2/2\lambda + (c^\alpha/\lambda)^2}$$

$$F_\gamma[u] := \sup_{\alpha\in\Lambda} \left[\gamma^\alpha (L^\alpha u - f^\alpha)\right]$$

Because $\gamma^\alpha > 0$, we have

$$F_\gamma[u] = 0 \iff \sup_{\alpha\in\Lambda}[L^\alpha u - f^\alpha] = 0. \tag{2}$$

The problem (HJB) for $u \in H^2(\Omega) \cap H^1_0(\Omega)$ is equivalent to

$$\mathcal{A}(u; v) := \int_\Omega F_\gamma[u]\, L_\lambda v \,\mathrm{d}x = 0 \quad \forall\, v \in H^2(\Omega) \cap H^1_0(\Omega), \tag{3}$$

where $L_\lambda v := \Delta v - \lambda v$.

2. PDE theory: proof of well-posedness

$$\textit{Notation:} \quad \|v\|_{H^2,\lambda}^2 := \int_\Omega |D^2 u|^2 + 2\lambda|\nabla u|^2 + \lambda^2|u|^2 \mathrm{d}x$$

Key ingredients

1. The Cordes condition, which implies that

$$\|F_\gamma[u] - F_\gamma[v] - L_\lambda(u-v)\|_{L^2} \leq \sqrt{1-\varepsilon}\|u-v\|_{H^2,\lambda} \tag{4}$$

2. Miranda–Talenti: for convex Ω, [Maugeri *et al.*, 2000]

$$\|w\|_{H^2,\lambda} \leq \|L_\lambda w\|_{L^2} \qquad \forall\, w \in H^2(\Omega) \cap H^1_0(\Omega) \tag{5}$$

3. The Browder–Minty theorem, a central result from nonlinear functional analysis. (*next slide*)

Browder–Minty Theorem: a general result from nonlinear functional analysis

Let X be a Banach space.

Let $\mathcal{A}\colon X \times X \to \mathbb{R}$ be linear in its second argument (only).

$\mathcal{A}$ is hemicontinuous if the mapping $[0,1] \ni t \mapsto \mathcal{A}\,(t\,u + (1-t)\,v; w)$ is continuous for any u, $v \in X$, uniformly over all $w \in X$.

$\mathcal{A}$ is strongly monotone if there exists a positive constant $c > 0$ such that

$$\frac{1}{c}\|u - v\|_X^2 \leq \mathcal{A}(u; u - v) - \mathcal{A}(v; u - v) \quad \forall\, u, v \in X. \tag{6}$$

Theorem (Browder–Minty)

Let X be a separable reflexive Banach space and let $\mathcal{A}\colon X \times X \to \mathbb{R}$ be a hemicontinuous and strongly monotone. Then, for each $\ell \in X^*$, there exists a unique $u \in X$ such that $\mathcal{A}(u; v) = \ell(v)$ for all $v \in X$.

Browder–Minty theorem is a generalisation of the Lax–Milgram theorem to nonlinear problems. [Renardy, 2004]

2. PDE theory: proof of well-posedness

Hemicontinuity is easier to check, so we show here *strong monotonicity*:

Recall $\mathcal{A}(u;v) = \int_\Omega F_\gamma[u] L_\lambda v \,\mathrm{d}x$.

$$\mathcal{A}(u;u-v) - \mathcal{A}(v;u-v) = \int_\Omega \left(F_\gamma[u] - F_\gamma[v]\right) L_\lambda(u-v)\,\mathrm{d}x.$$

Addition–subtraction of $\|L_\lambda(u-v)\|_{L^2}^2$ gives

$$\begin{aligned}\mathcal{A}(u;u-v) - \mathcal{A}(v;u-v) &= \|L_\lambda(u-v)\|_{L^2}^2 \\ &\quad + \int_\Omega \left(F_\gamma[u] - F_\gamma[v] - L_\lambda(u-v)\right) L_\lambda(u-v)\,\mathrm{d}x \\ &\geq (1-\sqrt{1-\varepsilon})\|L_\lambda(u-v)\|_{L^2}^2.\end{aligned}$$

Next: Use the inequalities (Cordes+Miranda–Talenti)

$\|F_\gamma[u] - F_\gamma[v] - L_\lambda(u-v)\|_{L^2} \leq \sqrt{1-\varepsilon}\|u-v\|_{H^2,\lambda} \leq \sqrt{1-\varepsilon}\|L_\lambda(u-v)\|_{L^2}$

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$$\mathcal{A}(u; u - v) - \mathcal{A}(v; u - v) \geq (1 - \sqrt{1 - \varepsilon})\|L_\lambda(u - v)\|_{L^2}^2$$

Use the Miranda–Talenti inequality

$$\|w\|_{H^2,\lambda} \leq \|L_\lambda w\|_{L^2} \quad \forall\, w \in H^2(\Omega) \cap H_0^1(\Omega)$$

to obtain strong monotonicity:

$$\|u - v\|_{H^2,\lambda}^2 \lesssim \mathcal{A}(u; u - v) - \mathcal{A}(v; u - v)$$

The existence and uniqueness result follows from the Browder–Minty Theorem. □

Approach to numerical analysis:

Since the proof of well-posedness hinges on the structure of

$$\mathcal{A}(u; v) = \int_\Omega F_\gamma[u] L_\lambda v \mathrm{d}x,$$

we will attempt to discretise the operator $\mathcal{A}$.

2. PDE Theory: summary

1. The Cordes condition originates from the PDE theory of nondivergence form elliptic equations with discontinuous coefficients.

2. The Cordes condition allows for a short proof of well-posedness for fully nonlinear HJB equations.

3. The PDE analysis suggests an approach to discretising the PDE.

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- *PDE Theory:* Analysis of HJB equations with Cordes coefficients.
- *Numerical methods:* High-order discontinuous Galerkin methods for HJB equations with Cordes coefficients.
    - Design of a consistent, stable and convergent method.
    - Error bounds.
    - Numerical experiments.

3. Numerics: overview

Our main results: *hp*-version DGFEM for HJB equations with *Cordes coefficients*

- The first provably consistent, stable and high order method for these problems.
- Error bounds for solutions with minimal regularity as well as high-order error bounds for more regular solutions.
- First numerical experiments demonstrating exponential convergence rates under *hp*-refinement for these problems.

I. S. & E. Süli, SINUM 2013: *Discontinuous Galerkin finite element approximation of nondivergence form elliptic equations with Cordes coefficients.*

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3. Numerics: overview

Further results:

- Superlinearly convergent semismooth Newton algorithm for the discretised nonlinear problem.
- h-version robust preconditioners for linearised systems.
- Extension of above results to parabolic equations as well as elliptic equations.

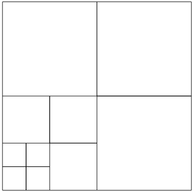
I. S. & E. Süli, accepted in Numerische Mathematik: *Discontinuous Galerkin finite element methods for time-dependent Hamilton–Jacobi–Bellman equations with Cordes coefficients.*

I. S., in review (arXiv:1409.4202): *Nonoverlapping domain decomposition preconditioners for discontinuous Galerkin finite element methods in H^2-type norms.*

3. Numerics: design of the method

Let $\{\mathcal{T}_h\}_h$ a shape-regular sequence of meshes on Ω.

- Shape-regular means no overly stretched elements (precise definition in e.g. [Brenner & Scott, 2003]).
- Elements composing the mesh can be parallelepipeds, simplices, or more generally any standard choice elements. Mixing and matching allowed!
- The mesh is *not assumed to be quasi-uniform* (very useful for *hp*-refinement).



Example of a shape-regular but non-uniform mesh

3. Numerics: design of the method

Construction of the discontinuous finite element space

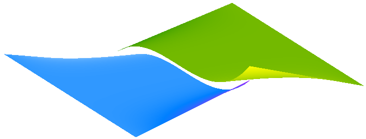
Choose a polynomial degree p_K = polynomial degree for each element K of the mesh $\mathcal{T}_h$.

Define the discontinuous finite element space as:

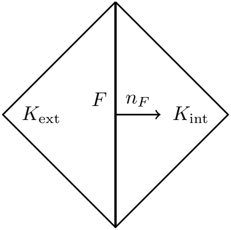
$$V_{h,\mathbf{p}} := \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_{p_K}(K) \; \forall K \in \mathcal{T}_h\}.$$

The set $\mathcal{P}_{p_K}(K)$ is either chosen as the space of piecewise polynomials of either *total* or *partial* degree p_K on the element K.

Since the method approximates functions in H^2-norms, we will use $p_K \geq 2$ for all elements K.



Notation of discontinuous Galerkin methods:



Distinguish interior and boundary faces

$$\mathcal{F}_h^i \text{ interior faces of } \mathcal{T}_h, \qquad \mathcal{F}_h^b \text{ boundary faces of } \mathcal{T}_h,$$

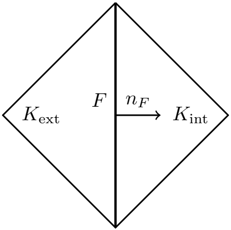
$$\mathcal{F}_h^{i,b} := \mathcal{F}_h^i \cup \mathcal{F}_h^b.$$

Jump operators over faces: if ϕ is either a piecewise continuous function or vector field, we define the jump $[\![\phi]\!]$ and the average $\{\phi\}$ by:

$$[\![\phi]\!] := \tau_F\left(\phi|_{K_{\text{ext}}}\right) - \tau_F\left(\phi|_{K_{\text{int}}}\right), \quad \{\phi\} := \tfrac{1}{2}\tau_F\left(\phi|_{K_{\text{ext}}}\right) + \tfrac{1}{2}\tau_F\left(\phi|_{K_{\text{int}}}\right), \quad \text{if } F \in \mathcal{F}_h^i,$$

$$[\![\phi]\!] := \tau_F\left(\phi|_{K_{\text{ext}}}\right), \qquad \{\phi\} := \tau_F\left(\phi|_{K_{\text{ext}}}\right), \qquad \text{if } F \in \mathcal{F}_h^b,$$

Notation of discontinuous Galerkin methods:



Let $\{t_i\}_{i=1}^{d-1} \subset \mathbb{R}^d$ be an orthonormal coordinate system on F. Define the *tangential gradient and divergence*

$$\nabla_{\mathrm{T}}\, u := \sum_{i=1}^{d-1} t_i \frac{\partial u}{\partial t_i}, \quad \mathrm{div}_{\mathrm{T}}\, \mathbf{v} := \sum_{i=1}^{d-1} \frac{\partial \mathbf{v}_i}{\partial t_i}. \tag{7}$$

3. Numerics: design of the method

The goal is to discretise

$$\mathcal{A}(u; v) = \int_\Omega F_\gamma[u]\, L_\lambda v \,\mathrm{d}x,$$

whilst conserving the strong monotonicity bound.

Recall main ingredients:

1. The Cordes condition.
2. Miranda–Talenti inequality: not conserved when replacing $H^2(\Omega) \cap H_0^1(\Omega)$ by $V_{h,\mathbf{p}}$.
3. The Browder–Minty theorem.

Solution:

- Miranda–Talenti inequality was derived from an integration by parts identity,
- We will *weakly enforce this identity* in the scheme (next slide)

Define the nonlinear form $A_h \colon V_{h,\mathbf{p}} \times V_{h,\mathbf{p}} \to \mathbb{R}$ by

$$A_h(u_h; v_h) := \sum_{K \in \mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h \rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K \in \mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h \rangle_K \right).$$

3. Numerics: design of the method

Define the nonlinear form $A_h \colon V_{h,\mathbf{p}} \times V_{h,\mathbf{p}} \to \mathbb{R}$ by

$$A_h(u_h; v_h) := \sum_{K \in \mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h \rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K \in \mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h \rangle_K \right).$$

$$\langle F_\gamma[u_h], L_\lambda v_h \rangle_K := \int_K \sup_{\alpha \in \Lambda} [\gamma^\alpha (L^\alpha u_h - f^\alpha)] \, (\Delta v_h - \lambda v_h) \, \mathrm{d}x.$$

3. Numerics: design of the method

Define the nonlinear form $A_h \colon V_{h,\mathbf{p}} \times V_{h,\mathbf{p}} \to \mathbb{R}$ by

$$A_h(u_h; v_h) := \sum_{K\in\mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h\rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K\in\mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h\rangle_K \right).$$

Jump penalisation with $\mu_F \simeq p_K^2/h_K$ and $\eta_F \simeq p_K^4/h_K^3$ for $F \subset \partial K$:

$$J_h(u_h, v_h) := \sum_{F\in\mathcal{F}_h^{i,b}} \left[\mu_F \langle [\![\nabla_{\mathrm{T}} u_h]\!], [\![\nabla_{\mathrm{T}} v_h]\!]\rangle_F + \eta_F \langle [\![u_h]\!], [\![v_h]\!]\rangle_F\right] + \sum_{F\in\mathcal{F}_h^{i}} \mu_F \langle [\![\nabla u_h \cdot n_F]\!], [\![\nabla v_h \cdot n_F]\!]\rangle_F .$$

3. Numerics: design of the method

Define the nonlinear form $A_h\colon V_{h,\mathbf{p}} \times V_{h,\mathbf{p}} \to \mathbb{R}$ by

$$A_h(u_h; v_h) := \sum_{K\in\mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h\rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K\in\mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h\rangle_K\right).$$

$$\langle L_\lambda u_h, L_\lambda v_h\rangle_K := \int_K (\Delta u_h - \lambda u_h)(\Delta v_h - \lambda v_h)\,\mathrm{d}x.$$

Define the nonlinear form $A_h\colon V_{h,\mathbf{p}} \times V_{h,\mathbf{p}} \to \mathbb{R}$ by

$$A_h(u_h; v_h) := \sum_{K\in\mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h\rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K\in\mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h\rangle_K \right).$$

$$B_h(u_h, v_h) := \sum_{K\in\mathcal{T}_h} \left[\langle D^2 u_h, D^2 v_h\rangle_K + 2\lambda\langle \nabla u_h, \nabla v_h\rangle_K + \lambda^2\langle u_h, v_h\rangle_K\right]$$
$$+ \sum_{F\in\mathcal{F}_h^i} \left[\langle \operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}}\{u_h\}, [\![\nabla v_h \cdot n_F]\!]\rangle_F + \langle \operatorname{div}_{\mathrm{T}} \nabla_{\mathrm{T}}\{v_h\}, [\![\nabla u_h \cdot n_F]\!]\rangle_F\right]$$
$$- \sum_{F\in\mathcal{F}_h^{i,b}} \left[\langle \nabla_{\mathrm{T}}\{\nabla u_h \cdot n_F\}, [\![\nabla_{\mathrm{T}} v_h]\!]\rangle_F + \langle \nabla_{\mathrm{T}}\{\nabla v_h \cdot n_F\}, [\![\nabla_{\mathrm{T}} u_h]\!]\rangle_F\right]$$
$$- \lambda \sum_{F\in\mathcal{F}_h^{i,b}} [\langle\{\nabla u_h \cdot n_F\}, [\![v_h]\!]\rangle_F + \langle\{\nabla v_h \cdot n_F\}, [\![u_h]\!]\rangle_F]$$
$$- \lambda \sum_{F\in\mathcal{F}_h^{i}} [\langle\{u_h\}, [\![\nabla v_h \cdot n_F]\!]\rangle_F + \langle\{v_h\}, [\![\nabla u_h \cdot n_F]\!]\rangle_F]$$

3. Numerics: design of the method

Define the nonlinear form $A_h\colon V_{h,\mathbf{p}} \times V_{h,\mathbf{p}} \to \mathbb{R}$ by

$$A_h(u_h; v_h) := \sum_{K\in\mathcal{T}_h} \langle F_\gamma[u_h], L_\lambda v_h\rangle_K + J_h(u_h, v_h) + \frac{1}{2}\left(B_h(u_h, v_h) - \sum_{K\in\mathcal{T}_h} \langle L_\lambda u_h, L_\lambda v_h\rangle_K\right).$$

Key consistency result: If $u \in H^2(\Omega) \cap H^1_0(\Omega)$ has well-defined second derivatives on faces F of the mesh, then

$$B_h(u, v_h) = \sum_K \langle L_\lambda u, L_\lambda v_h\rangle_K, \quad J_h(u, v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}.$$

Technical point: a sufficient condition is that $u \in H^s(K)$ with $s > 5/2$ for every $K \in \mathcal{T}_h$.

3. Numerics: consistency, stability and error bounds

Numerical scheme: find $u_h \in V_{h,\mathbf{p}}$ such that

$$A_h(u_h; v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}. \qquad \text{(scheme)}$$

Full theoretical justification given in [S. & Süli, SINUM 2014]:

- Consistency: sufficiently regular solution of (HJB) solves (scheme):

$$A_h(u; v_h) = 0 \quad \forall\, v_h \in V_{h,\mathbf{p}}.$$

- Stability: the nonlinear form A_h has a similar strong monotonicity bound as $\mathcal{A}$:

$$\|u_h - v_h\|_h^2 \lesssim A_h(u_h; u_h - v_h) - A_h(v_h; u_h - v_h) \quad \forall\, u_h,\, v_h \in V_{h,\mathbf{p}}.$$

  $\Longrightarrow$ existence & uniqueness of numerical solution, continuous dependence on data.

- Consistency+Stability $\Longrightarrow$ error bounds and convergence.

3. Numerics: error bounds

$$\|v_h\|_h^2 := \sum_{K\in\mathcal{T}_h}\left[|v_h|^2_{H^2(K)} + 2\lambda|v_h|^2_{H^1(K)} + \lambda^2\|v_h\|^2_{L^2(K)}\right] + J_h(v_h, v_h).$$

Theorem ([S. & Süli, SINUM 2014])

(Under previous assumptions & standard assumptions for DG meshes...)

Assume that $u \in H^{\mathbf{s}}(\Omega;\mathcal{T}_h)$*, with* $s_K > 5/2$ *for all* $K \in \mathcal{T}_h$*.*

$$\|u - u_h\|_h^2 \lesssim \sum_{K\in\mathcal{T}_h} \frac{h_K^{2t_K-4}}{p_K^{2s_K-5}}\|u\|^2_{H^{s_K}(K)},$$

where $t_K = \min(p_K + 1, s_K)$ *for each* $K \in \mathcal{T}_h$*.*

Simplified form:

$$\|u - u_h\| \lesssim \frac{h^{\min(s,p+1)-2}}{p^{s-5/2}}\|u\|_{H^s(\Omega)}.$$

- Quasi-optimal error bound.
- High-order convergence rates.
- Higher efficiency on well-chosen meshes.

If u has only minimal regularity, then we have the following quasi-optimal approximation property with respect to the H^2-conforming subspace:

Theorem

Under previous assumptions...
Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ *be the solution of* (HJB). *Then*

$$\|u - u_h\|_h \leq \inf_{z_h \in V_{h,\mathbf{p}} \cap H^2(\Omega) \cap H_0^1(\Omega)} \|u - z_h\|_h.$$

Interpretation: The DG method is at least as accurate, modulo constants, as any H^2-conforming method using the same mesh and same polynomial degrees.

Experiment 1 : Test of high order convergence rates

Example (Control of correlated diffusions)

Prototypical example of stochastic control from [S. & Süli, SINUM 2014]*:*

$$\mathrm{d}X_t = \underbrace{R^\top \begin{pmatrix} 1 & \sin\theta \\ 0 & \cos\theta \end{pmatrix}}_{\sigma(x,\alpha)} \begin{pmatrix} \mathrm{d}B_t^1 \\ \mathrm{d}B_t^2 \end{pmatrix}, \quad \alpha := (\theta, R) \in [0, \tfrac{\pi}{3}] \times \mathrm{SO}(2) =: \Lambda.$$

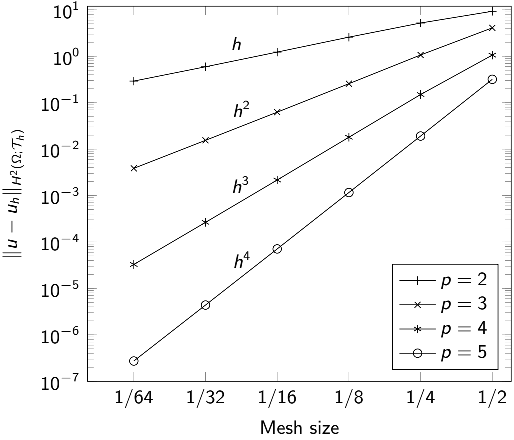
Then $a^\alpha := \frac{1}{2}\sigma\sigma^\top$ gives

$$a^\alpha = \frac{1}{2} R^\top \begin{pmatrix} 1+\sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{pmatrix} R$$

Principal difficulty: a^α becomes increasingly anisotropic as $\theta \to \pi/3$.

Example (Control of correlated diffusions)

Prototypical example of stochastic control from [S. & Süli, SINUM 2014]*:*
Uniform h-refinement on smooth solution $u(x,y) = \exp(xy)\sin(\pi x)\sin(\pi y)$:



Experiment 2: test of exponential convergence rates

Example (Strong anisotropy + boundary layer)

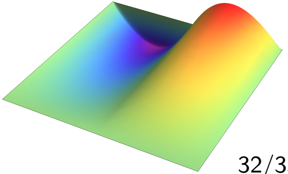
Let $\Omega = (0,1)^2$, $b^\alpha \equiv (0,1)$, $c^\alpha \equiv 10$ and define

$$a^\alpha := \alpha^\top \begin{pmatrix} 20 & 1 \\ 1 & 0.1 \end{pmatrix} \alpha, \quad \alpha \in \Lambda := \mathrm{SO}(2), \quad \lambda = \frac{1}{2}.$$

(Cordes) holds with $\varepsilon \approx 0.0024$ and $\lambda = 1/2$. Choose solution:

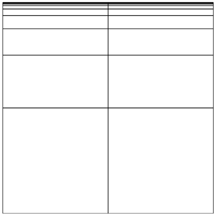
$$u(x,y) = (2x-1)\left(\mathrm{e}^{1-|2x-1|} - 1\right)\left(y + \frac{1-\mathrm{e}^{y/\delta}}{\mathrm{e}^{1/\delta}-1}\right), \quad \delta := 0.005 = \mathrm{O}(\varepsilon)$$

- Near-degenerate and anisotropic diffusion.
- Sharp boundary layer.
- Non-smooth solution.

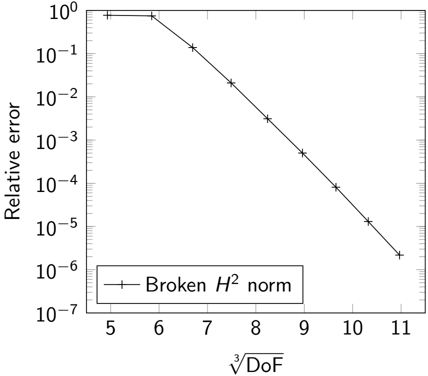


Example (Strong anisotropy + boundary layer)

We use boundary layer adapted meshes with p-refinement: $2 \leq p_K \leq 10$, from 100 to 1320 DoFs.



Boundary layer adapted mesh.



Exponential rate: $\|u - u_h\|_h \lesssim \exp(-c\sqrt[3]{\mathrm{DoF}})$.

3. Numerics: summary

1. Design of a consistent, stable and high order method for HJB equations with Cordes coefficients.
2. The central idea of the scheme is a weak enforcement of the Miranda–Talenti identity.
3. Error bounds for both regular and minimal regularity solutions.
4. Numerical experiments showing high order convergence, even exponential convergence rates.

What next?

Analysis:

- All error analysis so far is *a priori* analysis, i.e. the exact solution enters the error bounds.
- *A posteriori* analysis would be beneficial for adaptive algorithms.

Algorithms:

- Current preconditioners are robust with respect to h only.
- p-robust preconditioners for FEM/DGFEM in H^2 norms?

Thank you!