CNAc: Continuous Optimization
Problem set 1 — optimality conditions

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Hilary Term 2006, Dr Nick Gould

Instructions: Asterisked problems are intended as a homework assignment, while nonasterisked problems are not compulsory but can further help you understand the material. Please put your solutions in Denis Zuev's pigeon hole at the Maths Institute by 9AM on Monday of 3rd week.

***Problem 1.** Let $\mathcal{S} \subset \mathbb{R}^n$, let $f : S \rightarrow \mathbb{R}$, and suppose that $x + \alpha s \in \mathcal{S}$ for all $\alpha \in [0, 1]$.

(i) By defining $\theta(\alpha) = f(x + \alpha s)$ and using the integration-by-parts (Newton) formula

$$\theta(1) - \theta(0) = \int_0^1 \theta'(\alpha) d\alpha,$$

show that

$$|f(x+s) - f(x) - g(x)^T s| \leq \tfrac{1}{2}\gamma^L(x)\|s\|^2,$$

whenever $f \in C$ has a Lipschitz continuous gradient $g(x)$ (with Lipschitz constant $\gamma^L(x)$) within $\mathcal{S}$ (first result in Theorem 1.1).

(ii) Justify the integration-by-parts formula

$$\theta(1) - \theta(0) - \theta'(0) = \int_0^1 (1 - \alpha)\theta''(\alpha) d\alpha.$$

Hence show that

$$|f(x+s) - f(x) - g(x)^T s - \tfrac{1}{2}s^T H(x)s| \leq \tfrac{1}{6}\gamma^Q(x)\|s\|^3,$$

whenever $f \in C$ has a Lipschitz continuous Hessian $H(x)$ (with Lipschitz constant $\gamma^Q(x)$) within $\mathcal{S}$ (second result in Theorem 1.1).

***Problem 2.**

(i) Let $\mathcal{E}$, $\mathcal{A}$ be disjoint subsets of $\{1, \ldots, m\}$. Given any vectors g and a_i, $i \in \mathcal{E} \bigcup \mathcal{A}$, use Farkas' lemma to show that the set

$$\mathcal{S} = \{s \mid g^T s < 0, \;\; {a_i}^T s = 0 \text{ for } i \in \mathcal{E}, \;\text{ and }\; {a_i}^T s \geq 0 \text{ for } i \in \mathcal{A}\}$$

is empty if and only if

$$g \in C = \left\{ \sum_{i \in \mathcal{E}} z_i a_i + \sum_{i \in \mathcal{A}} y_i a_i \mid y_i \geq 0 \text{ for all } i \in \mathcal{A} \right\}.$$

(ii) Hence or otherwise deduce first-order necessary optimality conditions for the differentiable optimization problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \text{ subject to } c_i(x) = 0, i \in \mathcal{E}, \text{ and } c_i(x) \geq 0, i \in \mathcal{I}.$$

Problem 3. Suppose that $f_i(x)$, $i = 1, \ldots, m$, are twice-continuously differentiable functions of x. Consider the *non-differentiable* optimization problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) = \max_{1 \leq i \leq m} |f_i(x)|. \tag{1}$$

(i) Why might this problem be "non-differentiable"?

(ii) By arguing that (1) is equivalent to the *differentiable* problem

$$\underset{x \in \mathbb{R}^n u \in \mathbb{R}}{\text{minimize}} \quad u \text{ subject to } -u \leq f_i(x) \leq u$$

for some additional variable u, deduce first-order necessary optimality conditions for (1).

The "Method of Lagrange multipliers" is the direct application of the necessary and sufficient optimality conditions (Theorems 1.7–1.11) to solve constrained optimization problems. For the next three problems, you will gain experience in using this method.

***Problem 4†.** Use the method of Lagrange multipliers to solve the problem

$$\begin{array}{rl} \underset{x \in \mathbb{R}^2}{\text{minimize}} & \|x\| \\ \text{such that} & \left\| x - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| \geq 1 \\ \text{and} & \left\| x - \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\| \leq 1 \end{array} \tag{2}$$

where $\| \cdot \|$ denotes the (Euclidean) ℓ_2-norm.

***Problem 5†.** Consider the minimization problem

$$\min \ -0.1(x_1 - 4)^2 + x_2^2 \text{ such that } x_1^2 + x_2^2 - 1 \geq 0.$$

(i) Does this problem have a global minimiser?

(ii) Set up the KKT conditions for this problem.

(iii) Find all points x_* and vectors y_* of Lagrange multipliers so that (x_*, y_*) satisfy the KKT conditions.

(iv) Is the LICQ satisfied at theses x_*? [The linear-independence constraint qualification (LICQ) holds if the gradients of the active constraints are linearly independent.]

(v) Check if the sufficient optimality conditions hold at x_*.

Problem 6†. Consider the half space defined by $H = \{x \in \mathbb{R}^n : a^T x + b \geq 0\}$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ are given. Formulate and solve the optimisation problem of finding the point x in H that has the smallest Euclidean norm.

† Thanks to Raphael Hauser for these examples