# CNAc: Continuous Optimization
# Problem set 4 — linearly constrained optimization and penalty methods

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**Instructions:** Asterisked problems are intended as a homework assignment while the nonasterisked problem is not compulsory but can further help you understand the material. Please put your solutions in Denis Zuev's pigeon hole at the Maths Institute by 9AM on Monday of 7th week.

A set $\mathcal{C}$ is *convex* if for any two points $x, y \in \mathcal{C}$ $\alpha x + (1-\alpha)y \in \mathcal{C}$ for all $\alpha \in [0,1]$, i.e., all points on the line between $x$ and $y$ also lie in $\mathcal{C}$. A function $f$ is *convex* if its domain $\mathcal{C}$ is convex and

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

for all $\alpha \in [0,1]$, i.e., the value of $f$ at a point lying between $x$ and $y$ lies below the straight line joining the values $f(x)$ and $f(y)$. It is *strictly convex* if

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y)$$

for all $\alpha \in (0,1)$,

***Problem 1.**

(a) Show that if $x_*$ is a local minimizer of the convex function $f(x)$ over the convex set $\mathcal{C}$, then $x_*$ is a global minimizer.

(b) Show that the set of global minimizers of the convex function $f(x)$ is convex.

(c) Show that if $x_*$ is a local minimizer of the strictly convex function $f(x)$ over the convex set $\mathcal{C}$, then $x_*$ is the unique global minimizer.

(d) Suppose that $f(x) = g^T x + \frac{1}{2} x^T H x$. Show that $f(x)$ is convex if and only if $H$ is positive semi-definite.

(e) Suppose that $f(x)$ is as in part (d). Show that $f(x)$ is strictly convex if and only if $H$ is positive definite.

***Problem 2.**
Consider the equality-constrained quadratic program

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad g^T x + \tfrac{1}{2} x^T B x \ \text{ subject to } \ Ax = b,$$

where $g = -(1,1,1)^T$, $A = (1 \;\; 1 \;\; 0)$ and $b = 2$. Solve the problem when

(a)

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(b)

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and}$$

(c)

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

***Problem 3.**

Find the minimizer of $q(x) = \frac{1}{2}([x]_1 - 1)^2 + \frac{1}{2}([x]_2 - 0.5)^2$, where

$$\begin{aligned} [x]_1 + [x]_2 &\leq 1 && (1) \\ 3[x]_1 + [x]_2 &\leq 1.5 \qquad \text{and} && (2) \\ x &\geq 0, \end{aligned}$$

using the primal active-set method, starting from $x = (1/12, 25/28)$. [This is the example illustrated on the overheads in class.]

***Problem 4$^\dagger$.**

Consider the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad -x_1 - x_2 \text{ subject to } 1 - x_1^2 - x_2^2 = 0. \tag{3}$$

(a) Use the first-order necessary optimality (KKT) conditions to solve this problem.

(b) Let $x(\mu)$ be a local minimizer of the quadratic penalty function for (3). Show that $[x(\mu)]_1 = [x(\mu)]_2$ and $2[x(\mu)]_1^3 - [x(\mu)]_2 - \mu/2 = 0$.

(c) Among the two solutions for $x(\mu)$, pick the one for which $[x(\mu)]_1 > 0$. Show that as $\mu \to 0$,

$$[x(\mu)]_1 = \frac{1}{\sqrt{2}} + a\mu + O(\mu^2).$$

Find the constant $a$.

**Problem 5.**

Consider the *quartic* penalty function

$$\Phi(x, \mu) = f(x) + \frac{1}{4\mu}\|c(x)\|_2^4$$

for the equality-constrained minimization problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \text{ subject to } c(x) = 0, \tag{4}$$

where $f$, $c \in \mathcal{C}^2$, Suppose that

$$[y_k]_i \stackrel{\text{def}}{=} -\frac{\|c(x_k)\|_2^2 [c(x_k)]_i}{\mu_k},$$

that

$$\|\nabla_x \Phi(x_k, \mu_k)\|_2 \leq \epsilon_k,$$

where $\epsilon_k$ converges to zero as $k \to \infty$, and that $x_k$ converges to $x_*$ for which $A(x_*)$ is full rank. Show that $x_*$ satisfies the first-order necessary optimality conditions for the problem (4) and $\{y_k\}$ converge to the associated Lagrange multipliers $y_*$.

† Thanks to Raphael Hauser (and indirectly Roger Fletcher) for this example.