CNAc: Continuous Optimization
Problem set 6 — SQP methods

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Instructions: Asterisked problems are intended as a homework assignment. Please put your solutions in Denis Zuev's pigeon hole at the Maths Institute by 9AM on Monday of 1st week of Trinity Term.

***Problem 1.**

(a) Suppose that $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has a Lipschitz continuous Jacobian within some open set $\mathcal{S}$ containing a root x_* of the nonlinear equation $F(x) = 0$. Show that the *Newton* sequence $\{x_k\}$, where $x_{k+1} = x_k - (\nabla_x F(x_k))^{-1} F(x_k)$, converges Q-quadratically to x_* from any starting point sufficiently close to x_* provided that $\nabla_x F(x_*)$ is non-singular. [Hint: use Theorem 1.3].

(b) Consider the nonlinear equation $F(x) = x^2 = 0$. What is the Q-rate of convergence of Newton's method? How does this reflect on the result from part (a)?

***Problem 2.**

(a) What is the solution of the problem $2(x_1^2 + x_2^2 - 1) - x_1$ for which $x_1^2 + x_2^2 - 1 = 0$? Show that its Lagrange multiplier is $3/2$?

(b) Compute the SQP step s for this problem from the feasible point $x_k = (\cos\theta, \sin\theta)^T$ $(\theta \in (-\pi, \pi))$, using the (optimal) Lagrange multiplier estimate y_*. Show that both the objective function and the constraint violation would increase if the SQP step were taken if $\theta \neq 0$ [This is the Maratos effect).

(c) Show that the second-order correction

$$\begin{pmatrix} I & A^T(x) \\ A(x) & 0 \end{pmatrix} \begin{pmatrix} s^{\mathrm{C}} \\ -y^{\mathrm{C}} \end{pmatrix} = - \begin{pmatrix} 0 \\ c(x+s) \end{pmatrix}$$

to s is small relative to s. [You might also show that adding s^{C} to s gives a step that reduces the non-differentiable penalty function, but this is messy so I don't insist!].

***Problem 3.**

Show that the smallest (in the Euclidean norm) s that satisfies $As + c = 0$ may be found by solving the linear system

$$\begin{pmatrix} I & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} s \\ -y \end{pmatrix} = - \begin{pmatrix} 0 \\ c \end{pmatrix}$$

involving an auxiliary vector y.

***Problem 4.**

Formulate the ℓ_∞QP subproblem with an ℓ_1-norm trust region

$$\underset{s\in\mathbb{R}^n}{\text{minimize}} \;\; g_k^T s + \tfrac{1}{2}s^T B_k s + \rho\|c_k + A_k s\|_\infty \;\text{ subject to }\; \|s\|_1 \leq \Delta_k$$

as a quadratic program.

***Problem 5.**

Consider the quartic penalty function

$$\Phi(x,\mu) = f(x) + \frac{1}{4\mu}\|c(x)\|_2^4$$

we examined in problem sheet 4. Show that the equivalent version of Theorem 7.1 holds for this Φ, namely that the SQP search direction is a descent direction when B_k is positive definite, whenever

$$\mu_k \leq \frac{\|c(x_k)\|_2^3}{\|y_{k+1}\|_2}.$$