SECTION C: CONTINUOUS OPTIMISATION
PROBLEM SET 7

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***Problem 1.** Consider the problem

$$\begin{aligned} &\min -x_1x_2x_3 \\ &\text{s.t.} \quad 72 - x_1 - 2x_2 - 2x_3 = 0. \end{aligned} \tag{0.1}$$

(i) For $x^* = [\,24\ 12\ 12\,]^{\mathrm{T}}$ verify that there exists a Lagrange multiplier λ^* such that (x^*, λ^*) is a KKT point.

(ii) Now let

$$x(\mu) := \arg\min_{x \in \mathbb{R}^2} Q(x, \mu),$$

where $Q(x, \mu)$ is the quadratic penalty function for (0.1). Verify that the explicit expression for $x(\mu)$ given by

$$x_1(\mu) = 2x_2(\mu), \quad x_2(\mu) = x_3(\mu) = \frac{24}{1 + \sqrt{1 - 8\mu}}$$

satisfies $\nabla_x Q(x(\mu), \mu) = 0$, and verify that $x(\mu) \to x^*$ as $\mu \to 0$.

(iii) Let $\mu = 1/9$. Find $x(\mu)$ and verify that $D^2_{xx} Q(x(\mu), \mu)$ is positive definite, so that $x(\mu)$ is a strict local minimiser of $Q(x, \mu)$.

(iv) Show that $-g(x(\mu))/\mu \to \lambda^*$, where g is the equality constraint function in (0.1).

***Problem 2.** Consider the problem

$$\begin{aligned} &\min -x_1 - x_2 \\ &\text{s.t.} \quad 1 - x_1^2 - x_2^2 = 0. \end{aligned} \tag{0.2}$$

(i) Solve this problem using the method of Lagrange multipliers.

(ii) Now let $x(\mu)$ be a local minimiser of

$$\min_{x \in \mathbb{R}^2} Q(x, \mu), \tag{0.3}$$

where Q is the quadratic penalty function for (0.2). Show that $x_1(\mu) = x_2(\mu)$ and $2x_1^3(\mu) - x_1(\mu) - \mu/2 = 0$.

(iii) Among the two solutions for $x(\mu)$, pick the one for which $x_1(\mu) > 0$. Show that as $\mu \to 0$,

$$x_1 = \frac{1}{\sqrt{2}} + a\mu + O(\mu^2).$$

Find the constant a.

(iv) Now consider the problem

$$\begin{aligned} &\min -x_1 - x_2 \\ \text{s.t.}\quad &1 - x_1^2 - x_2^2 = 0, \\ &x_2 - x_1^2 \geq 0. \end{aligned}$$

Show how the penalty function may be modified to solve this problem. Show that there is a range of values of μ for which the minimisers of the two penalty functions agree.

***Problem 3.** The quadratic penalty function method was applied to the problem

$$\begin{aligned} &\min -x_1 - x_2 + x_3 \\ \text{s.t.}\quad &0 \leq x_3 \leq 1, \\ &x_1^3 + x_3 \leq 1, \\ &x_1^2 + x_2^2 + x_3^2 \leq 1, \end{aligned}$$

and the following data was obtained, where k counts the iterations:

k	μ_k	$x_1(\mu_k)$	$x_2(\mu_k)$	$x_3(\mu_k)$
1	1	0.834379	0.834379	-0.454846
2	0.1	0.728324	0.728324	-0.087920
3	0.01	0.709557	0.709557	-0.009864
4	0.001	0.707356	0.707356	-0.001017

(i) Use the table to estimate the optimal solution x^*, the optimal vector of Lagrange multipliers λ^*, and the set of active constraints at optimality $\mathcal{A}(x^*)$.
(ii) Check that (x^*, λ^*) is a KKT point.
(iii) What accuracy is predicted by the theory for $x(\mu_4)$ as an approximation of x^*? Check if the predicted accuracy is consistent with the data obtained, and thus, either accept or reject your hunch about x^*.