

SECTION C: CONTINUOUS OPTIMISATION
REVISION CLASS 1, PART 2

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Problem 1 Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x) = x_1x_2$ where $x = (x_1, x_2)$, and consider the trust region $B_\Delta(\hat{x}) = \{x : \|x - \hat{x}\| \leq \Delta\}$ around $\hat{x} = (1, 1/2)$, where $\|\cdot\|$ denotes the Euclidean norm. We are interested in solving the trust-region subproblem

$$(TR) \quad \min_{x \in B_\Delta(\hat{x})} f(x).$$

- (i) Show that the Cauchy point is $y^c = (3/8, -3/4)$.
- (ii) Find an explicit parameterisation of the dogleg path.
- (iii) Show that the objective function is decreasing on the first segment of the dogleg path and increasing on the second segment. Why does this phenomenon occur?
- (iv) Conclude that there exists a critical value $\Delta_c > 0$ such that the dogleg solution to (TR) is given by the Cauchy point for all $\Delta \geq \Delta_c$.
- (v) Apply Steihaug's method with exact line-searches and find a formula for its solution $y^s(\Delta)$ as a function of Δ .
- (vi) Show that $f(y^s(\Delta))$ is strictly decreasing as a function of Δ .

Problem 2 Consider the BFGS updating formula

$$B_{k+1} = B_k - \frac{B_k \delta_k \delta_k^T B_k}{\delta_k^T B_k \delta_k} + \frac{\gamma_k \gamma_k^T}{\gamma_k^T \delta_k}$$

- (i) Explain the meaning of each of the terms appearing in this formula and explain their connection with the so-called "secant-condition".

Now let f be a strictly convex quadratic function in $x \in \mathbb{R}^n$,

$$f(x) = \frac{1}{2}x^T A x + b^T x + c,$$

where $b \in \mathbb{R}^n$ is a vector and $A \in \mathbb{R}^{n \times n}$ a symmetric positive definite matrix. Starting with $B_0 = I$, we will apply BFGS steps with exact line searches.

- (ii) Let S_k be the linear subspace of \mathbb{R}^n such that $s \in S_k$ if $B_k s = f''(x_k)s$ and $\nabla f(x_k)^T s = 0$. Show that if $s \in S_k$ then $\delta_k^T B_k s$ and $\gamma_k s$ are both zero.
- (iii) Conclude from this that $s \in S_k$ implies $\nabla f(x_{k+1})^T s = 0$.
- (iv) Show that $\delta_k^T B_k s = \gamma_k^T s = 0$ implies $B_{k+1}s = B_k s$ and conclude that

$$S_0 \subseteq S_1 \subseteq S_2 \subseteq \dots \subseteq S_{k+1}.$$

- (v) Show that δ_k is not in S_k .
- (vi) Show that $\delta_k \in S_{k+1}$, and conclude that $\dim S_k < \dim S_{k+1}$.
- (vii) Assuming that we stop the procedure when $\nabla f(x_k) = 0$, show that the algorithm terminates after at most n steps.