SECTIONS C: CONTINUOUS OPTIMISATION
REVISION CLASS 2, PART 1

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Problem 1. Consider the following nonlinear programming problem,

$$\begin{aligned} \min\ f(x) &= \frac{x_1^2}{4} + x_2^2 \\ \text{s.t. } g_1(x) &= 4 - x_1 \geq 0, \\ g_2(x) &= x_1^2 + x_2^2 - 25 \geq 0. \end{aligned} \tag{0.1}$$

(i) [3 pts] Use the Bolzano-Weierstrass theorem to prove that if $f : K \to \mathbb{R}$ is a continuous function defined on the compact set K then there exists a global minimiser $x^* \in K$ of the optimisation problem $\min_{x\in K} f(x)$.

(ii) [4 pts] Sketch the feasible domain of problem (0.1). Also sketch the set $\{x \in \mathbb{R}^2 : f(x) \leq f(-10, 0)\}$. Using part (i) argue that (0.1) has a global minimiser.

(iii) [4 pts] Write down the Lagrangian $\mathcal{L}$, its x-gradient $\nabla_x \mathcal{L}$ and the KKT conditions for problem (0.1).

(iv) [4 pts] For $x^* = \left[\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right]$ and $\widehat{x} = \left[\begin{smallmatrix} 0 \\ 5 \end{smallmatrix}\right]$ find Lagrange multiplier vectors λ^* and $\widehat{\lambda}$ such that (x^*, λ^*) and $(\widehat{x}, \widehat{\lambda})$ satisfy the KKT conditions.

(v) [4 pts] Write down necessary and sufficient conditions that characterise when $d \in \mathbb{R}^2$ is a feasible exit direction from x^*, and use these conditions to prove that x^* is a local minimiser of (0.1).

(vi) [6 pts] Use the same type of argument to prove that $\widehat{x}$ is not a local minimiser.

Problem 2. Consider the unconstrained minimisation problem

$$\min_{x\in\mathbb{R}^2} f(x) = \frac{(x_1 + x_2 - 10)^2}{\kappa} + (x_2 - x_1)^2, \tag{0.2}$$

where $\kappa > 1$ is a fixed parameter.

(i) [4 pts] Starting from $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$, apply an exact line search along the search direction $e_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and show that the updated point is

$$x^+ = \begin{bmatrix} \frac{\kappa-1}{\kappa+1} x_2 + \frac{10}{1+\kappa} \\ x_2 \end{bmatrix}.$$

(ii) [1 pt] Now take $e_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ as search direction and show that if an exact line search is applied to $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$, then

$$x^+ = \begin{bmatrix} x_1 \\ \frac{\kappa-1}{\kappa+1} x_1 + \frac{10}{1+\kappa} \end{bmatrix}.$$

(iii) [5 pts] Let $x^{[0]}$ be the origin and consider the sequence $(x^{[k]})_{\mathbb{N}}$ defined iteratively as follows: $x^{[k+1]}$ is obtained by applying an exact line search at $x^{[k]}$ with search direction e_i where

$$i = \begin{cases} 1 & (k = 0, 2, 4, \dots) \\ 2 & (k = 1, 3, 5, \dots). \end{cases}$$

Show that

$$x^{[k]} = \begin{cases} \frac{10}{\kappa-1} \begin{bmatrix} \sum_{i=1}^{k-1} \lambda^i \\ \sum_{i=1}^{k} \lambda^i \end{bmatrix}, & (k = 2, 4, 6, \dots), \\ \\ \frac{10}{\kappa-1} \begin{bmatrix} \sum_{i=1}^{k} \lambda^i \\ \sum_{i=1}^{k-1} \lambda^i \end{bmatrix}, & (k = 3, 5, \dots), \end{cases}$$

where $\lambda = \frac{\kappa-1}{\kappa+1}$.

(iv) [4 pts] Compute $x^* = \lim_{k\to\infty} x^{[k]}$ and, using sufficient optimality conditions, show that x^* is the global optimiser of f.

(v) [6 pts] Define what it means for a sequence to be Q-linearly convergent, and show that $x^{[k]}$ converges to x^* Q-linearly.

(vi) [5 pts] What happens to the convergence factor for large κ? What happens when κ is close to 1? How is κ related to the condition number of $f''(x^*)$?