CNAc: Continuous Optimization
Solutions to problem set 5 — interior-point methods

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Problem 1.

(a) The sequence has limit zero and

$$\sigma_{k+1}/\sigma_k^q = (\log(k+1))^q/\log(k+2) = (\log(k+1)/\log(k+2))(\log(k+1))^{q-1},$$

which is Q-sublinear for all $q < 1$.

(b) Again the sequence has limit zero and

$$\sigma_{k+1}/\sigma_k^q = 2^{-k-1}/2^{-qk} = 2^{k(q-1)-1}$$

which diverges for $q > 1$, and for $q = 1$, $\kappa = 2^{-1} < 1$. Thus the convergence is Q-linear.

(c) Once more the sequence has limit zero and

$$\sigma_{k+1}/\sigma_k^q = 2^{-(k+1)^2}/2^{-qk^2} = 2^{(q-1)k^2-2k-1}$$

which again diverges if $q > 1$. But when $q = 1$, $\sigma_{k+1}/\sigma_k^q = 2^{-2k-1}$ which converges to zero, and thus the convergence is Q-superlinear.

(d) And again, the sequence has limit zero and

$$\sigma_{k+1}/\sigma_k^q = 2^{-2^(k+1)}/2^{-q2^k} = 2^{-2^k(2-q)}.$$

In this case the convergence is Q-superlinear with any Q-factor $q \leq 2$, i.e., Q-quadratic.

Problem 2.
Differentiating $\Phi(x,\mu)$ gives

$$\nabla_x \Phi(x,\mu) = g(x) - \sum_{i=1}^{m} \frac{\mu}{c_i^2(x)} a_i(x) \quad (1)$$

which suggests that

$$y_i(x) = \frac{\mu}{c_i^2(x)} \quad (2)$$

as Lagrange multiplier estimates.

The proof of the theorem only changes in a few places. The bound (1) on the norm of the inactive Lagrange multiplier estimates becomes

$$\|(y_k)_{\mathcal{I}}\|_2 \leq 2\mu_k \sqrt{|\mathcal{I}|}/\min_{i\in\mathcal{I}} |c_i^2(x_*)|.$$

The same argument then shows that $y_k \rightarrow y_*$, and hence from (1) that $g(x_*) - A^T(x_*)y_* = 0$. But then (2) gives $y_k > 0$ and $[y_k]_i c_i^2(x_k) = \mu_k$ and hence $y_* \geq 0$ and $[y_*]_i c_i^2(x_*) = 0$. Thus either $[y_*]_i = 0$ or $c_i(x_*) = 0$ and all of the first-order necessary optimality conditions hold.

Problem 3.

(a) The barrier function is

$$\Phi(x,\mu) = \frac{1}{1+x^2} - \mu \log x.$$

Let ω be any desired number. When $x > 1$, $\Phi(x) \leq 1/2 - \mu \log x < \omega$. for all $x > x_\omega = e^{(1-\omega)/\mu}$. Thus Φ is unbounded from below for any $\mu > 0$.

(b) The barrier function is

$$\Phi(x,\mu) = \tfrac{1}{2}x^2 - \mu \log(x - 2a)$$

from which we deduce that $x(\mu) - y(\mu) = 0$ where $y(\mu) = \mu/(x(\mu) - a)$. Hence

$$x(\mu) = a + \sqrt{a^2 + \mu} \;\text{ and }\; y(\mu) = \frac{\mu}{\sqrt{a^2+\mu} - a}.$$

Since $x_* = 2a$ and $y_* = 2a$,

$$x(\mu) - x_* = \sqrt{a^2+\mu} - a = a(\sqrt{1+\mu/a^2} - 1).$$

But as $1 + \frac{1}{4}\mu/a^2 \leq \sqrt{1+\mu/a^2} \leq 1 + \mu/a^2$ for all $0 \leq \mu/a^2 \leq 8$, we have

$$\tfrac{1}{4}\mu/a \leq |x(\mu) - x_*| \leq \mu/a$$

which depends linearly on μ. Thus the Q-rate of convergence is linear as a function of μ so long as $a > 0$. Likewise

$$|y(\mu) - y_*| \leq \mu/a,$$

and thus the Lagrange multiplier estimates converge Q-linearly as a function of μ.

(c) The barrier function is

$$\Phi(x,\mu) = \tfrac{1}{2}x^2 - \mu \log x$$

from which we have $x(\mu) = \mu^{\frac{1}{2}} = y(\mu)$. But $x_* = 0 = y_*$, so

$$x(\mu) - x_* = \mu^{\frac{1}{2}} \;\text{ and }\; y(\mu) - y_* = \mu^{\frac{1}{2}}.$$

The Q-rate of convergence is sub-linear as a function of μ.