

# Part 1: Optimality conditions and why they are important

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$$c(x) \geq 0, \quad g(x) + A^T(x)y = 0, \quad y \geq 0$$

MSc course on nonlinear optimization

# OPTIMIZATION PROBLEMS

## Unconstrained minimization:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x)$$

where the **objective function**  $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

## Equality constrained minimization:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \quad \text{subject to} \quad c(x) = 0$$

where the **constraints**  $c : \mathbb{R}^n \longrightarrow \mathbb{R}^m$  ( $m \leq n$ )

## Inequality constrained minimization:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \quad \text{subject to} \quad c(x) \geq 0$$

where  $c : \mathbb{R}^n \longrightarrow \mathbb{R}^m$  ( $m$  may be larger than  $n$ )

# NOTATION

Use the following throughout the course:

$$g(x) \stackrel{\text{def}}{=} \nabla_x f(x) \quad \text{gradient of } f$$

$$H(x) \stackrel{\text{def}}{=} \nabla_{xx} f(x) \quad \text{Hessian matrix of } f$$

$$a_i(x) \stackrel{\text{def}}{=} \nabla_x c_i(x) \quad \text{gradient of } i\text{th constraint}$$

$$H_i(x) \stackrel{\text{def}}{=} \nabla_{xx} c_i(x) \quad \text{Hessian of } i\text{th constraint}$$

$$A(x) \stackrel{\text{def}}{=} \nabla_x c(x) \equiv \begin{pmatrix} a_1^T(x) \\ \dots \\ a_m^T(x) \end{pmatrix} \quad \text{Jacobian matrix of } c$$

$$\ell(x, y) \stackrel{\text{def}}{=} f(x) - y^T c(x) \quad \text{Lagrangian function, where } y \text{ are Lagrange multipliers}$$

$$H(x, y) \stackrel{\text{def}}{=} \nabla_{xx} \ell(x, y) \equiv H(x) - \sum_{i=1}^m y_i H_i(x) \quad \text{Hessian of the Lagrangian}$$

# LIPSCHITZ CONTINUITY

- $\mathcal{X}$  and  $\mathcal{Y}$  open sets
- $F : \mathcal{X} \rightarrow \mathcal{Y}$
- $\|\cdot\|_{\mathcal{X}}$  and  $\|\cdot\|_{\mathcal{Y}}$  are norms

Then

- $F$  is **Lipschitz continuous at**  $x \in \mathcal{X}$  if  $\exists \gamma(x)$  such that

$$\|F(z) - F(x)\|_{\mathcal{Y}} \leq \gamma(x) \|z - x\|_{\mathcal{X}}$$

for all  $z \in \mathcal{X}$ .

- $F$  is **Lipschitz continuous throughout/in**  $\mathcal{X}$  if  $\exists \gamma$  such that

$$\|F(z) - F(x)\|_{\mathcal{Y}} \leq \gamma \|z - x\|_{\mathcal{X}}$$

for all  $x$  and  $z \in \mathcal{X}$ .

## USEFUL TAYLOR APPROXIMATIONS

**Theorem 1.1.** Let  $\mathcal{S}$  be an open subset of  $\mathbb{R}^n$ , and suppose  $f : \mathcal{S} \rightarrow \mathbb{R}$  is continuously differentiable throughout  $\mathcal{S}$ . Suppose further that  $g(x)$  is Lipschitz continuous at  $x$ , with Lipschitz constant  $\gamma^L(x)$  in some appropriate vector norm. Then, if the segment  $x + \theta s \in \mathcal{S}$  for all  $\theta \in [0, 1]$ ,

$$|f(x + s) - m^L(x + s)| \leq \tfrac{1}{2} \gamma^L(x) \|s\|^2, \quad \text{where}$$

$$m^L(x + s) = f(x) + g(x)^T s.$$

If  $f$  is twice continuously differentiable throughout  $\mathcal{S}$  and  $H(x)$  is Lipschitz continuous at  $x$ , with Lipschitz constant  $\gamma^Q(x)$ ,

$$|f(x + s) - m^Q(x + s)| \leq \tfrac{1}{6} \gamma^Q(x) \|s\|^3, \quad \text{where}$$

$$m^Q(x + s) = f(x) + g(x)^T s + \tfrac{1}{2} s^T H(x) s.$$

## MEAN VALUE THEOREM

**Theorem 1.2.** Let  $\mathcal{S}$  be an open subset of  $\mathbb{R}^n$ , and suppose  $f : \mathcal{S} \rightarrow \mathbb{R}$  is twice continuously differentiable throughout  $\mathcal{S}$ . Suppose further that  $s \neq 0$ , and that the interval  $[x, x + s] \in \mathcal{S}$ . Then

$$f(x + s) = f(x) + g(x)^T s + \frac{1}{2} s^T H(z) s$$

for some  $z \in (x, x + s)$ .

## ANOTHER USEFUL TAYLOR APPROXIMATION

**Theorem 1.3.** Let  $\mathcal{S}$  be an open subset of  $\mathbb{R}^n$ , and suppose  $F : \mathcal{S} \rightarrow \mathbb{R}^m$  is continuously differentiable throughout  $\mathcal{S}$ . Suppose further that  $\nabla_x F(x)$  is Lipschitz continuous at  $x$ , with Lipschitz constant  $\gamma^L(x)$  in some appropriate vector norm and its induced matrix norm. Then, if the segment  $x + \theta s \in \mathcal{S}$  for all  $\theta \in [0, 1]$ ,

$$\|F(x + s) - M^L(x + s)\| \leq \tfrac{1}{2}\gamma^L(x)\|s\|^2,$$

where

$$M^L(x + s) = F(x) + \nabla_x F(x)s$$

# OPTIMALITY CONDITIONS

Optimality conditions are useful because:

- they provide a means of guaranteeing that a candidate solution is indeed optimal (**sufficient conditions**), and
- they indicate when a point is not optimal (**necessary conditions**)

Furthermore they

- guide in the design of algorithms, since lack of optimality  $\iff$  indication of improvement



# UNCONSTRAINED MINIMIZATION

## First-order necessary optimality:

**Theorem 1.4.** Suppose that  $f \in C^1$ , and that  $x_*$  is a local minimizer of  $f(x)$ . Then

$$g(x_*) = 0.$$

## Second-order necessary optimality:

**Theorem 1.5.** Suppose that  $f \in C^2$ , and that  $x_*$  is a local minimizer of  $f(x)$ . Then  $g(x_*) = 0$  and  $H(x_*)$  is positive semi-definite, that is

$$s^T H(x_*) s \geq 0 \text{ for all } s \in \mathbb{R}^n.$$

## PROOF OF THEOREM 1.4

Suppose otherwise, that  $g(x_*) \neq 0$ .

Taylor expansion in the direction  $-g(x_*)$  gives

$$f(x_* - \alpha g(x_*)) = f(x_*) - \alpha \|g(x_*)\|^2 + O(\alpha^2).$$

For sufficiently small  $\alpha$ ,  $\frac{1}{2}\alpha \|g(x_*)\|^2 \geq O(\alpha^2)$ , and thus

$$f(x_* - \alpha g(x_*)) \leq f(x_*) - \frac{1}{2}\alpha \|g(x_*)\|^2 < f(x_*).$$

This contradicts hypothesis that  $x_*$  is a local minimizer.

## PROOF OF THEOREM 1.5

Suppose otherwise that  $s^T H(x_*)s < 0$ .

Taylor expansion in the direction  $s$  gives

$$f(x_* + \alpha s) = f(x_*) + \frac{1}{2}\alpha^2 s^T H(x_*)s + O(\alpha^3),$$

since  $g(x_*) = 0$ . For sufficiently small  $\alpha$ ,  $-\frac{1}{4}\alpha^2 s^T H(x_*)s \geq O(\alpha^3)$ , and thus

$$f(x_* + \alpha s) \leq f(x_*) + \frac{1}{4}\alpha^2 s^T H(x_*)s < f(x_*).$$

This contradicts hypothesis that  $x_*$  is a local minimizer.

## UNCONSTRAINED MINIMIZATION (cont.)

### Second-order sufficient optimality:

**Theorem 1.6.** Suppose that  $f \in C^2$ , that  $x_*$  satisfies the condition  $g(x_*) = 0$ , and that additionally  $H(x_*)$  is positive definite, that is

$$s^T H(x_*) s > 0 \text{ for all } s \neq 0 \in \mathbb{R}^n.$$

Then  $x_*$  is an isolated local minimizer of  $f$ .

## PROOF OF THEOREM 1.6

Continuity  $\implies H(x)$  positive definite  $\forall x$  in open ball  $\mathcal{N}$  around  $x_*$ .

$x_* + s \in \mathcal{N}$  + generalized mean value theorem  $\implies \exists z$  between  $x_*$  and  $x_* + s$  for which

$$\begin{aligned} f(x_* + s) &= f(x_*) + g(x_*)^T s + \tfrac{1}{2} s^T H(z) s \\ &= f(x_*) + \tfrac{1}{2} s^T H(z) s \\ &> f(x_*) \end{aligned}$$

$\forall s \neq 0 \implies x_*$  is an isolated local minimizer.

# EQUALITY CONSTRAINED MINIMIZATION

## First-order necessary optimality:

**Theorem 1.7.** Suppose that  $f$ ,  $c \in C^1$ , and that  $x_*$  is a local minimizer of  $f(x)$  subject to  $c(x) = 0$ . Then, so long as a first-order constraint qualification holds, there exist a vector of Lagrange multipliers  $y_*$  such that

$$\begin{aligned} c(x_*) &= 0 \quad (\text{primal feasibility}) \text{ and} \\ g(x_*) - A^T(x_*)y_* &= 0 \quad (\text{dual feasibility}). \end{aligned}$$

## PROOF OF THEOREM 1.7

Constraint qualification  $\implies \exists$  vector valued  $C^2$  ( $C^3$  for Theorem 1.8) function  $x(\alpha)$  of the scalar  $\alpha$  for which

$$x(0) = x_* \quad \text{and} \quad c(x(\alpha)) = 0$$

and

$$x(\alpha) = x_* + \alpha s + \tfrac{1}{2}\alpha^2 p + O(\alpha^3)$$

+ Taylor's theorem  $\implies$

$$\begin{aligned} 0 &= c_i(x(\alpha)) = c(x_* + \alpha s + \tfrac{1}{2}\alpha^2 p + O(\alpha^3)) \\ &= c_i(x_*) + a_i^T(x_*) (\alpha s + \tfrac{1}{2}\alpha^2 p) + \tfrac{1}{2}\alpha^2 s^T H_i(x_*) s + O(\alpha^3) \\ &= \alpha a_i^T(x_*) s + \tfrac{1}{2}\alpha^2 (a_i^T(x_*) p + s^T H_i(x_*) s) + O(\alpha^3) \end{aligned}$$

Matching similar asymptotic terms  $\implies$

$$A(x_*) s = 0 \tag{1}$$

and

$$a_i^T(x_*) p + s^T H_i(x_*) s = 0 \quad \forall i = 1, \dots, m \tag{2}$$

Now consider objective function

$$\begin{aligned}
f(x(\alpha)) &= f(x_* + \alpha s + \tfrac{1}{2}\alpha^2 p + O(\alpha^3)) \\
&= f(x_*) + g(x_*)^T (\alpha s + \tfrac{1}{2}\alpha^2 p) + \tfrac{1}{2}\alpha^2 s^T H(x_*) s + O(\alpha^3) \\
&= f(x_*) + \alpha g(x_*)^T s + \tfrac{1}{2}\alpha^2 (g(x_*)^T p + s^T H(x_*) s) + O(\alpha^3)
\end{aligned} \tag{3}$$

$f(x)$  unconstrained along  $x(\alpha) \implies$

$$g(x_*)^T s = 0 \text{ for all } s \text{ such that } A(x_*)s = 0. \tag{4}$$

Let  $N$  be a basis for null space of  $A(x_*) \implies$

$$g(x_*) = A^T(x_*)y_* + Nz_* \tag{5}$$

for some  $y_*$  and  $z_*$ . (4)  $\implies g^T(x_*)N = 0 + A(x_*)N = 0 \implies$

$$0 = N^T g(x_*) = N^T A^T(x_*)y_* + N^T Nz_* = N^T Nz_*.$$

$\implies N^T Nz_* = 0 + N$  full rank  $\implies z_* = 0 + (5) \implies$

$$g(x_*) - A^T(x_*)y_* = 0.$$



## EQUALITY CONSTRAINED MINIMIZATION (cont.)

### Second-order necessary optimality:

**Theorem 1.8.** Suppose that  $f$ ,  $c \in C^2$ , and that  $x_*$  is a local minimizer of  $f(x)$  subject to  $c(x) = 0$ . Then, provided that first- and second-order constraint qualifications hold, there exist a vector of Lagrange multipliers  $y_*$  such that

$$s^T H(x_*, y_*) s \geq 0 \text{ for all } s \in \mathcal{N}$$

where

$$\mathcal{N} = \{s \in \mathbb{R}^n \mid A(x_*)s = 0\}$$

## PROOF OF THEOREM 1.8

$$g(x_*) - A^T(x_*)y_* = 0. \quad (6)$$

while (3)  $\implies$

$$f(x(\alpha)) = f(x_*) + \tfrac{1}{2}\alpha^2 (p^T g(x_*) + s^T H(x_*)s) + O(\alpha^3) \quad (7)$$

for all  $s$  and  $p$  satisfying  $A(x_*)s = 0$  and

$$a_i^T(x_*)p + s^T H_i(x_*)s = 0 \quad \forall i = 1, \dots, m. \quad (8)$$

Hence, necessarily,

$$p^T g(x_*) + s^T H(x_*)s \geq 0 \quad (9)$$

But (6) + (8)  $\implies$

$$p^T g(x_*) = \sum_{i=1}^m (y_*)_i p^T a_i(x_*) = - \sum_{i=1}^m (y_*)_i s^T H_i(x_*)s$$

$\implies$  (9) is equivalent to

$$s^T \left( H(x_*) - \sum_{i=1}^m (y_*)_i H_i(x_*) \right) s \equiv s^T H(x_*, y_*)s \geq 0$$

for all  $s$  satisfying  $A(x_*)s = 0$ .

# INEQUALITY CONSTRAINED MINIMIZATION

## First-order necessary optimality:

**Theorem 1.9.** Suppose that  $f$ ,  $c \in C^1$ , and that  $x_*$  is a local minimizer of  $f(x)$  subject to  $c(x) \geq 0$ . Then, provided that a first-order constraint qualification holds, there exist a vector of Lagrange multipliers  $y_*$  such that

$$\begin{aligned} c(x_*) &\geq 0 && \text{(primal feasibility),} \\ g(x_*) - A^T(x_*)y_* &= 0 && \text{(dual feasibility) and} \\ \text{and } y_* &\geq 0 \\ c_i(x_*)[y_*]_i &= 0 && \text{(complementary slackness).} \end{aligned}$$

## PROOF OF THEOREM 1.9

Consider feasible perturbations about  $x_*$ .  $c_i(x_*) > 0 \implies c_i(x) > 0$  for small perturbations  $\implies$  need only consider perturbations that are constrained by  $c_i(x) \geq 0$  for  $i \in \mathcal{A} \stackrel{\text{def}}{=} \{i : c_i(x_*) = 0\}$ .

Consider  $x(\alpha)$ :  $x(0) = x_*$ ,  $c_i(x(\alpha)) \geq 0$  for  $i \in \mathcal{A}$  and

$$x(\alpha) = x_* + \alpha s + \tfrac{1}{2}\alpha^2 p + O(\alpha^3)$$

$\implies$

$$\begin{aligned} 0 &\leq c_i(x(\alpha)) = c(x_* + \alpha s + \tfrac{1}{2}\alpha^2 p + O(\alpha^3)) \\ &= c_i(x_*) + a_i(x_*)^T \alpha s + \tfrac{1}{2}\alpha^2 p + \tfrac{1}{2}\alpha^2 s^T H_i(x_*) s + O(\alpha^3) \\ &= \alpha a_i(x_*)^T s + \tfrac{1}{2}\alpha^2 \left( a_i(x_*)^T p + s^T H_i(x_*) s \right) + O(\alpha^3) \end{aligned}$$

$\forall i \in \mathcal{A} \implies$

$$s^T a_i(x_*) \geq 0 \quad \forall i \in \mathcal{A} \tag{10}$$

and

$$p^T a_i(x_*) + s^T H_i(x_*) s \geq 0 \quad \text{when } s^T a_i(x_*) = 0 \quad \forall i \in \mathcal{A} \tag{11}$$

Expansion (3) of  $f(x(\alpha))$

$$f(x(\alpha)) = f(x_*) + \alpha g(x_*)^T s + \frac{1}{2}\alpha^2 (g(x_*)^T p + s^T H(x_*)s) + O(\alpha^3)$$

$\implies x_*$  can only be a local minimizer if

$$\mathcal{S} = \{s \mid s^T g(x_*) < 0 \text{ and } s^T a_i(x_*) \geq 0 \text{ for } i \in \mathcal{A}\} = \emptyset.$$

Result then follows directly from Farkas' Lemma:

**Farkas' Lemma.** Given any vectors  $g$  and  $a_i, i \in \mathcal{A}$ , the set

$$\mathcal{S} = \{s \mid s^T g < 0 \text{ and } s^T a_i \geq 0 \text{ for } i \in \mathcal{A}\}$$

is empty if and only if

$$g = \sum_{i \in \mathcal{A}} y_i a_i$$

for some  $y_i \geq 0, i \in \mathcal{A}$

## INEQUALITY CONSTRAINED MINIMIZATION (cont.)

### Second-order necessary optimality:

**Theorem 1.10.** Suppose that  $f$ ,  $c \in C^2$ , and that  $x_*$  is a local minimizer of  $f(x)$  subject to  $c(x) \geq 0$ . Then, provided that first- and second-order constraint qualifications hold, there exist a vector of Lagrange multipliers  $y_*$  for which primal/dual feasibility and complementary slackness requirements hold as well as

$$s^T H(x_*, y_*) s \geq 0 \quad \text{for all } s \in \mathcal{N}_+$$

where

$$\mathcal{N}_+ = \left\{ s \in \mathbb{R}^n \left| \begin{array}{l} s^T a_i(x_*) = 0 \text{ if } c_i(x_*) = 0 \text{ \& } [y_*]_i > 0 \\ s^T a_i(x_*) \geq 0 \text{ if } c_i(x_*) = 0 \text{ \& } [y_*]_i = 0 \end{array} \right. \right\}.$$

# PROOF OF THEOREM 1.10

Expansion

$$f(x(\alpha)) = f(x_*) + \alpha g(x_*)^T s + \frac{1}{2} \alpha^2 (g(x_*)^T p + s^T H(x_*) s) + O(\alpha^3)$$

for change in objective function dominated by  $\alpha s^T g(x_*)$  for feasible perturbations unless  $s^T g(x_*) = 0$ , in which case the expansion

$$f(x(\alpha)) = f(x_*) + \frac{1}{2} \alpha^2 (p^T g(x_*) + s^T H(x_*) s) + O(\alpha^3)$$

is relevant  $\implies$

$$p^T g(x_*) + s^T H(x_*) s \geq 0 \tag{12}$$

holds for all feasible  $s$  for which  $s^T g(x_*) = 0 \implies$

$$0 = s^T g(x_*) = \sum_{i \in \mathcal{A}} (y_*)_i s^T a_i(x_*) \implies \text{either } (y_*)_i = 0 \text{ or } a_i(x_*)^T s = 0.$$

$\implies$  second-order feasible perturbations characterised by  $s \in \mathcal{N}_+$ .

Focus on *subset* of all feasible arcs that ensure  $c_i(x(\alpha)) = 0$  if  $(y_*)_i > 0$  and  $c_i(x(\alpha)) \geq 0$  if  $(y_*)_i = 0$  for  $i \in \mathcal{A} \implies s \in \mathcal{N}_+$ .

When  $c_i(x(\alpha)) = 0 \implies$

$$a_i^T(x_*)p + s^T H_i(x_*)s = 0$$

$\implies$

$$\begin{aligned} p^T g(x_*) &= \sum_{i \in \mathcal{A}} (y_*)_i p^T a_i(x_*) = \sum_{\substack{i \in \mathcal{A} \\ (y_*)_i > 0}} (y_*)_i p^T a_i(x_*) \\ &= - \sum_{\substack{i \in \mathcal{A} \\ (y_*)_i > 0}} (y_*)_i s^T H_i(x_*)s = - \sum_{i \in \mathcal{A}} (y_*)_i s^T H_i(x_*)s \end{aligned}$$

$$\begin{aligned} + (12) \implies s^T H(x_*, y_*)s &\equiv s^T \left( H(x_*) - \sum_{i=1}^m (y_*)_i H_i(x_*) \right) s \\ &= p^T g(x_*) + s^T H(x_*)s \geq 0. \end{aligned}$$

for all  $s \in \mathcal{N}_+$



## INEQUALITY CONSTRAINED MINIMIZATION (cont.)

### Second-order sufficient optimality:

**Theorem 1.11.** Suppose that  $f$ ,  $c \in C^2$ , that  $x_*$  and a vector of Lagrange multipliers  $y_*$  satisfy

$$c(x_*) \geq 0, g(x_*) - A^T(x_*)y_* = 0, y_* \geq 0, \text{ and } c_i(x_*)[y_*]_i = 0$$

and that

$$s^T H(x_*, y_*) s > 0$$

for all  $s$  in the set

$$\mathcal{N}_+ = \left\{ s \in \mathbb{R}^n \left| \begin{array}{l} s^T a_i(x_*) = 0 \text{ if } c_i(x_*) = 0 \text{ \& } [y_*]_i > 0 \text{ \& } \\ s^T a_i(x_*) \geq 0 \text{ if } c_i(x_*) = 0 \text{ \& } [y_*]_i = 0. \end{array} \right. \right\}.$$

Then  $x_*$  is an isolated local minimizer of  $f(x)$  subject to  $c(x) \geq 0$ .

## PROOF OF THEOREM 1.11

Consider any feasible arc  $x(\alpha)$ . Already shown

$$s^T a_i(x_*) \geq 0 \quad \forall i \in \mathcal{A} \quad (13)$$

and

$$p^T a_i(x_*) + s^T H_i(x_*)s \geq 0 \quad \text{when} \quad s^T a_i(x_*) = 0 \quad \forall i \in \mathcal{A} \quad (14)$$

and that second-order feasible perturbations are characterized by  $\mathcal{N}_+$ .

$$\begin{aligned} (14) \implies p^T g(x_*) &= \sum_{i \in \mathcal{A}} (y_*)_i p^T a_i(x_*) = \sum_{i \in \mathcal{A}} (y_*)_i p^T a_i(x_*) \\ &\quad \overset{s^T a_i(x_*)=0}{=} - \sum_{i \in \mathcal{A}} (y_*)_i s^T H_i(x_*)s = - \sum_{i \in \mathcal{A}} (y_*)_i s^T H_i(x_*)s, \end{aligned}$$

and hence by assumption that

$$\begin{aligned} p^T g(x_*) + s^T H(x_*)s &\geq s^T \left( H(x_*) - \sum_{i=1}^m (y_*)_i H_i(x_*) \right) s \\ &\equiv s^T H(x_*, y_*)s > 0 \end{aligned}$$

$\forall s \in \mathcal{N}_+ + (3) + (13) \implies f(x(\alpha)) > f(x_*) \quad \forall$  sufficiently small  $\alpha$ .