C10: CONTINUOUS OPTIMISATION
PROBLEM SET 1

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Instructions: Asterisked problems are intended as a homework assignment, while nonasterisked problems are not compulsory but can further help you understand the material. Please hand in solutions by Monday noon and clearly mark them for "Nicolas Jeannequin". Hand in at the reception of the Comlab.

***Problem 1.** Let $f : D \to \mathbb{R}$ be a C^2 function defined on a convex open domain $D \subset \mathbb{R}^n$.

(i) Let $u \in \mathbb{R}^n$ and $x \in D$. Write down the second order Taylor development of the function $t \mapsto f(x + tu)$ around $t = 0$.

(ii) Using Theorem 2.4 (ii) of Lecture 1 and the first part of this exercise, show that if f is convex then $u^{\mathrm{T}} H(x)u \geq 0$. Conclude from this that $H(x)$ is positive semidefinite.

(iii) Show that

$$f(x + tu) = f(x) + t\nabla f(x)^{\mathrm{T}} u + t^2 \int_0^1 \int_0^1 \vartheta u^{\mathrm{T}} H(x + \tau\vartheta tu) u d\tau d\vartheta.$$

(iv) Using Theorem 2.4 (ii) and part (iii) of this problem, show that if $H(y)$ is positive semidefinite for all $y \in D$ then f is convex.

(v) Using the same approach as in part (iv), show that if $H(y)$ is positive definite for all $y \in D$ then f is *strictly* convex.

***Problem 2.** Let C be a positive definite symmetric $n \times n$ matrix, $b \in \mathbb{R}^n$ a vector and $a \in \mathbb{R}$ a constant, and let us consider the function

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$x \mapsto a + b^{\mathrm{T}} x + \frac{1}{2} x^{\mathrm{T}} C x.$$

(i) Show that f is strictly convex.

(ii) Let $x_k, d_k \in \mathbb{R}^n$ with $d_k \neq 0$ and consider the function $\phi : \alpha \mapsto f(x_k + \alpha d_k)$, defined for $\alpha \in \mathbb{R}$. Show that ϕ is a strictly convex quadratic polynomial in α.

(iii) Compute the global minimiser α^* of ϕ.

(iv) Show that if d_k is a descent direction, then α^* satisfies the Wolfe conditions for any $c_1 \in (0, 1/2)$ and $c_2 \in (c_1, 1)$.

***Problem 3.** Let Algorithm 3.2 from Lecture 2 be applied to a C^1 function f with Λ-Lipschitz continuous gradient, and let the step size α_k be computed according to the rule

$$\alpha_k := \inf\{\alpha \geq 0 : \phi'(\alpha) = 0\},$$

that is, the algorithm uses so-called *exact line searches*. Suppose $\alpha_k \in (0, \infty)$ and $d_k \neq 0$ is a descent direction.

(i) Show that

$$f(x_{k+1}) \leq f(x_k) + \int_0^\beta \phi'(\alpha) d\alpha \qquad \forall \beta \in [0, \alpha_k].$$

(ii) Using the Cauchy-Schwartz inequality and the Λ-Lipschitz continuity of the gradient, show that

$$\phi'(\alpha) \leq \nabla f(x_k)^{\mathrm{T}} d_k + \alpha \Lambda \|d_k\|^2.$$

(iii) Use parts (i) and (ii) to show that

$$f(x_{k+1}) \leq f(x_k) - \frac{(\cos^2 \theta_k)\|\nabla f(x_k)\|^2}{2\Lambda},$$

where θ_k is the angle between d_k and $-\nabla f(x_k)$.