SECTIONS C: CONTINUOUS OPTIMISATION
REVISION CLASS 1, PART 1

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Problem 1 (The three parts of this problem are unrelated to one another.)

(i) Show that the Lagrangian dual of the problem

$$\begin{aligned}&\min \frac{1}{2}\sigma x_1^2 + \frac{1}{2}x_2^2 + x_1,\\ &\text{s.t. } x_1 \geq 0\end{aligned}$$

is a maximisation problem in terms of a Lagrange multiplier λ. For the cases $\sigma = +1$ and $\sigma = -1$, investigate whether the local solution of the dual gives the multiplier λ^* which exists at the local solution to the primal, and explain the difference between the two cases.

(ii) Consider the problem

$$\begin{aligned}&\min_{x\in\mathbb{R}} f(x) = 0,\\ &\text{s.t. } - e^x \geq 0.\end{aligned}$$

Verify that the constraint is concave but inconsistent, so that the feasible region is empty. Set up the Lagrangian dual problem and show that it is solved by $\lambda = 0$ and any x.

(iii) Consider finding the KKT points of the problem

$$\begin{aligned}&\max \frac{1}{3}\sum_{i=1}^{n} x_i^3,\\ &\text{s.t. } \sum_{i=1}^{n} x_i = 0,\\ &\qquad \sum_{i=1}^{n} x_i^2 = n\end{aligned}$$

for any $n \geq 2$. Use the method of Lagrange multipliers (with multipliers λ and μ respectively) to determine the general form of a KKT point for the problem (for general n). For any given n, identify a KKT point $(x_1^*, \ldots, x_n^*)$ with $x_1^* > 0$ and $x_2^*, \ldots, x_n^* < 0$. By examining second order sufficient conditions, show that this point is a local maximiser.

Problem 2 Consider finding the stationary point x^* of a given quadratic function $q(x)$, of which the Hessian matrix G is nonsingular and has only one negative eigenvalue. Let s be a given direction of negative curvature $s^{\mathrm{T}}Gs < 0$. Let $x^{(1)}$ be a given point, and let $x^{(2)}$ maximise $q(x^{(1)} + \alpha s)$ over α.

(i) If Z is a given $n \times (n-1)$ matrix with independent columns, such that $Z^{\mathrm{T}}Gs = 0$, write down the set of points X such that $x - x^{(2)}$ is G-conjugate to s, that is, $s^{\mathrm{T}}G(x - x^{(2)}) = 0$.

(ii) It can be shown that the matrix $S = (s, Z)$ is nonsingular and by Sylvester's Law that $S^{\mathrm{T}}GS$ has just one negative eigenvalue. Use these results (without proving them) to show that $Z^{\mathrm{T}}GZ$ is positive definite and consequently that

$$\begin{aligned} &\min q(x) \\ &\text{s.t. } x \in X \end{aligned}$$

has a unique minimiser x^*. Express x^* in terms of $x^{(2)}$ and $g^{(2)} = \nabla q(x^{(2)})$, and verify that x^* is also the unique saddle point of $q(x)$ in $\mathbb{R}^n$.

(iii) Show that a suitable Z matrix can be obtained from an elementary Householder orthogonal matrix $Q = \mathrm{I} - 2ww^{\mathrm{T}}$, where w is a unit vector such that $Q\gamma = \pm\|\gamma\|_2 e_1$, where $\gamma = Gs$, and where e_1 is the first column of the identity matrix I.