SECTION C: CONTINUOUS OPTIMISATION
PROBLEM SET 5

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***Problem 1.** Consider the general nonlinear programming problem (NLP). We say that the Mangasarian-Fromowitz constraint qualification (MFCQ) holds if there exists a vector $w \in \mathbb{R}^n$ such that

$$\begin{aligned} &\nabla g_i(x^*)^{\mathrm{T}} w > 0 \quad \big(i \in \mathcal{A}(x^*)\big), \\ &\nabla g_i(x^*)^{\mathrm{T}} w = 0 \quad (i \in \mathcal{E}), \\ &\{\nabla g_i(x^*) : i \in \mathcal{E}\} \quad \text{linearly independent.} \end{aligned}$$

Show that for the feasible region defined by

$$\begin{aligned} (x_1 - 1)^2 + (x_2 - 1)^2 &\leq 2, \\ (x_1 - 1)^2 + (x_2 + 1)^2 &\leq 2, \\ x_1 &\geq 0, \end{aligned}$$

the MFCQ is satisfied at $x^* = (0, 0)$ but the LICQ is not satisfied.

***Problem 2.** Recall that the Bolzano-Weierstrass theorem says that any sequence $(x_k)_{\mathbb{N}} \subset K$ of points from a compact set K has a convergent subsequence $(x_{k_i})_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} x_{k_i} \in K$.

(i) Use the Bolzano-Weierstrass theorem to prove that if $f : K \to \mathbb{R}$ is a continuous function defined on the compact set K then there exists a global minimiser $x^* \in K$ of the optimization problem $\min_{x \in K} f(x)$.

(ii) Consider the following nonlinear programming problem,

$$\begin{aligned} &\min\ x_1^2 + x_2^2 \\ &\text{s.t.}\ -2x_1 - x_2 + 10 \leq 0, \\ &\qquad -x_1 \leq 0. \end{aligned} \tag{0.1}$$

Using part (i), prove that a global minimiser of (0.1) exists.

(ii) Find the global minimiser by use of the method of Lagrange multipliers.

***Problem 3.** Let us revisit the proof of Lemma 2.3 from Lecture 9 and analyse what happens if the LICQ is replaced by the MFCQ.

(i) Show that if d satisfies Conditions (2.2) of Lecture 9 and if w is as in Problem 1, then for every $\delta > 0$ the vector $d + \delta w$ also satisfies (2.2).

(ii) Mimicking the proof of Lemma 2.3 with $d + \delta w$ in place of d and neglecting all inequality constraints, show that there exists $\tilde{\epsilon} > 0$ and a path $x \in C^k\big((-\tilde{\epsilon}, \tilde{\epsilon}), \mathbb{R}^n\big)$ such that

$$\begin{aligned} x(0) &= x^*, \\ \frac{d}{dt}x(0) &= d + \delta w, \\ g_i(x(t)) &= t\nabla g_i(x^*)^{\mathrm{T}}(d + \delta w), \qquad (i \in \mathcal{E}). \end{aligned}$$

(iii) Show that there exists $\epsilon \in (0, \tilde{\epsilon})$ such that $g_j(x(t)) \geq 0$ for $(j \in \mathcal{I}, t \in [0, \epsilon))$.

***Problem 4.** Consider the minimisation problem

$$\begin{aligned} \min \ & -0.1(x_1 - 4)^2 + x_2^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 - 1 \geq 0. \end{aligned}$$

(i) Does this problem have a global minimiser?
(ii) Find (x^*, λ^*) for which the KKT conditions are satisfied.
(iii) Is the LICQ satisfied at x^*?
(iv) Characterise the set of feasible exit directions from x^*.
(v) Check that the sufficient optimality conditions hold at x^* to show that x^* is a local minimiser.