SECTION C: CONTINUOUS OPTIMISATION
PROBLEM SET 5: SOLUTIONS

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Solution to Problem 1: Let $g_1(x_1, x_2) = 2 - (x_1 - 1)^2 - (x_2 - 1)^2$, $g_2(x_1, x_2) = 2 - (x_1 - 1)^2 - (x_2 + 1)^2$, and $g_3(x_1, x_2) = x_1$. Then all three constraints $g_i(x_1, x_2) \geq 0$ $(i = 1, 2, 3)$ are active at $(0, 0)$. Moreover, $\nabla g_1(0, 0) = \binom{2}{2}$, $\nabla g_2(0, 0) = \binom{2}{-2}$, and $\nabla g_3(0, 0) = \binom{1}{0}$ are linearly dependent. However, $w = \binom{1}{0}$ satisfies $w^{\mathrm{T}}\nabla g_1(0, 0) = w^{\mathrm{T}}\nabla g_2(0, 0) = 2w^{\mathrm{T}}\nabla g_3(0, 0) = 2 > 0$, and there are no equality constraints. Therefore the Mangasarian–Fromowitz constraint is satisfied.

Solution to Problem 2: (i) Let $f_{\min} = \inf\{f(x) : x \in K\}$. Then there exists a sequence of points $(x_k)_{\mathbb{N}} \subset K$ such that

$$f(x_k) \leq f_{\min} + \frac{1}{k}, \qquad (k \in \mathbb{N}).$$

The Bolzano-Weierstrass theorem implies that there exists a subsequence $(x_{k_i})_{\mathbb{N}}$ such that $x^* := \lim_{i\to\infty} x_{k_i} \in K$. Since f is continuous, this implies

$$f_{\min} \leq f(x^*) = \lim_{i\to\infty} f(x_{k_i}) \leq f_{\min} + \lim_{i\to\infty} \frac{1}{k} = f_{\min},$$

and hence, $f(x^*) = f_{\min}$.

(ii) We have $f(x_1, x_2) = x_1^2 + x_2^2 \geq 0$ on $\mathbb{R}^2$. Moreover, the feasible domain is closed and $(5, 0)$ is feasible with finite objective value 25. Therefore, the minimiser can be searched in the intersection of the closed disk of radius 5 with the feasible domain. This intersection is a closed and bounded subset of $\mathbb{R}^n$, and hence it is also compact. It now follows from part (i) that f achieves its global minimum at some point in this compact set.

(iii) The constraints can be rewritten in standard form as $g_1(x_1, x_2) = 2x_1 + x_2 - 10 \geq 0$ and $g_2(x_1, x_2) = x_1 \geq 0$. The Lagrangian of this problem is

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = x_1^2 + x_2^2 - \lambda_1(2x_1 + x_2 - 10) - \lambda_2 x_1.$$

The KKT conditions are the following:

$$2x_1 - 2\lambda_1 - \lambda_2 = 0, \tag{0.1}$$
$$2x_2 - \lambda_1 = 0, \tag{0.2}$$
$$2x_1 + x_2 - 10 \geq 0, \tag{0.3}$$
$$x_1 \geq 0, \tag{0.4}$$
$$\lambda_1(2x_1 + x_2 - 10) = 0, \tag{0.5}$$
$$\lambda_2 x_1 = 0, \tag{0.6}$$
$$\lambda_1 \geq 0, \tag{0.7}$$
$$\lambda_2 \geq 0. \tag{0.8}$$

If $\lambda_1 = 0$, then it follows from (0.1) that $2x_1 = \lambda_2$, and then (0.6) implies that $x_1 = 0$. Moreover, (0.2) implies that $x_2 = 0$. But $g_1(0,0) < 0$, and hence this is not a solution. Therefore, $\lambda_1 > 0$ and by (0.5), g_1 is active. If $x_1 = 0$ then (0.1) implies that $\lambda_2 = -2\lambda_1 < 0$, contradicting (0.8). Therefore, $x_1 > 0$ and then $\lambda_2 = 0$ because of (0.6). By (0.1), $x_1 = \lambda_1$, and then $x_2 = x_1/2$ because of (0.2). Since g_1 is active, it follows that $(x_1, x_2) = (4, 2)$. Since this is the only point satisfying the KKT conditions, and since the LICQ holds everywhere on the boundary of the feasible domain and in particular at the minimiser of the problem, $(x_1, x_2) = (4, 2)$ must be the solution of the problem.

Solution to Problem 3: (i) Note that

$$\begin{aligned} (d + \delta w) &\neq 0, \\ (d + \delta w)^{\mathrm{T}} \nabla g_i(x^*) &= 0 \quad (i \in \mathcal{E}), \\ (d + \delta w)^{\mathrm{T}} \nabla g_j(x^*) &> 0 \quad (j \in \mathcal{A}(x^*)), \end{aligned} \tag{0.9}$$

the first inequality following from the last set of inequalities. Therefore, $(d + \delta w)$ satisfies the required conditions.

(ii) Since $\{\nabla g_i(x^*) : i \in \mathcal{E}\}$ are linearly independent, there exists $Z \in \mathbb{R}^{(n-p)\times n}$ such that

$$M = \begin{bmatrix} g'_{\mathcal{E}}(x^*) \\ Z \end{bmatrix}$$

is a nonsingular $n \times n$ matrix. Let $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be defined by

$$(x, t) \mapsto \begin{bmatrix} g_{\mathcal{E}}(x) - t g'_{\mathcal{E}}(x^*)(d + \delta w) \\ Z(x - x^* - t(d + \delta w)) \end{bmatrix}.$$

Then

$$h'(x^*, 0) = \begin{bmatrix} M & -M(d + \delta w) \end{bmatrix}$$

and $D_x h(x^*, 0) = M$ is nonsingular. The Implicit Function Theorem implies that there exists a map $x \in C^k\big((-\tilde{\epsilon}, \tilde{\epsilon}), \mathbb{R}^n\big)$ and a neighbourhood $\mathcal{V}(x^*)$ of x^* such that for $(x, t) \in \mathcal{V}(x^*) \times (-\tilde{\epsilon}, \tilde{\epsilon})$ we have

$$h(x, t) = 0 \Leftrightarrow x = x(t).$$

The claim now follows as in the proof of Lemma 2.3.

(iii) The same argument as in Lemma 2.3 shows that there exists $\bar{\epsilon} \in (0, \epsilon)$ such that $g_j(x(t)) > 0$ for all $t \in (-\bar{\epsilon}, \bar{\epsilon})$. If $j \in \mathcal{A}(x^*)$, then Taylor's theorem shows

$$g_j(x(t)) = g_j(x^*) + t\nabla g_j(x^*)^{\mathrm{T}}(d + \delta w) + O(t^2).$$

But since $\nabla g_j(x^*)^{\mathrm{T}}(d + \delta w) > 0$, there exists $t_j \in (0, \tilde{t})$ such that the right-hand side is strictly larger than $g_j(x^*) = 0$ for all $t \in (0, t_j)$. It suffices now to choose $\epsilon \leq \min(\{t_j : j \in \mathcal{A}(x^*)\} \cup \{\bar{\epsilon}\})$.

Solution to Problem 4. (i) The objective function is unbounded along the line $x_2 = 0$, $x_1 \to \infty$. Thus, no global solution exists, but we can find a local minimum with the method of Lagrange multipliers.

(ii) We have

$$\begin{aligned} \nabla_x \mathcal{L}(x, \lambda) &= \begin{bmatrix} -0.2(x_1 - 4) - 2\lambda x_1 \\ 2x_2 - 2\lambda x_2 \end{bmatrix}, \\ \nabla_{xx} \mathcal{L}(x, \lambda) &= \begin{bmatrix} -0.2 - 2\lambda & 0 \\ 0 & 2 - 2\lambda \end{bmatrix}. \end{aligned}$$

The point $x^* = (1, 0)^{\mathrm{T}}$ satisfies the KKT conditions with $\lambda^* = 0.3$.

(iii) The active set at x^* is $\mathcal{A}(x^*) = \{1\}$, and since $\nabla g_1(x^*) = (2, 0)^{\mathrm{T}} \neq 0$, the LICQ is satisfied.

(iv) Since the LICQ is satisfied, Conditions (1.3) from Lecture 10 are an exact characterisation of the feasible exit directions from x^*. The set of vectors that satisfy these conditions is $\{d \in \mathbb{R}^2 : 2d_1 \geq 0, d_1^2 + d_2^2 > 0\}$.

(v) The set of feasible exit directions that also satisfy Equation $\lambda_j^* d^{\mathrm{T}} \nabla g_j(x^*) = 0$ for all $j \in \mathcal{A}(x^*)$ is $\{d \in \mathbb{R}^2 : d_1 = 0, d_2 \neq 0\}$. For any d from this set we have

$$d^{\mathrm{T}} \nabla_{xx} \mathcal{L}(x^*, \lambda^*) d = \begin{bmatrix} 0 & d_2 \end{bmatrix} \begin{bmatrix} -0.4 & 0 \\ 0 & 1.4 \end{bmatrix} \begin{bmatrix} 0 \\ d_2 \end{bmatrix} = 1.4 d_2^2 > 0.$$

Therefore, the second order sufficient optimality conditions are satisfied and x^* is a local minimiser our problem.