SECTION C: CONTINUOUS OPTIMISATION
PROBLEM SET 6

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***Problem 1.** Consider the nonlinear optimisation problem

$$\begin{aligned} \text{(CP)} \qquad & \min_{x\in\mathbb{R}^n} f(x) \\ & \text{s.t.} \quad Ax = b \\ & \qquad\quad g_{\mathrm{I}}(x) \geq 0, \end{aligned}$$

where f and $-g_j$ ($j \in \mathcal{E}$) are convex functions and $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, that is, (CP) is a convex programming problem. The Slater constraint qualification is satisfied if the row vectors of A are linearly independent and there exists $z \in \mathbb{R}^n$ such that $Az = b$ and $g_{\mathcal{I}}(z) > 0$.

(i) For $j \in \mathcal{I}$ and $x \in \mathbb{R}^n$ fixed, show that

$$\{z \in \mathbb{R}^n : g_j(z) > g_j(x)\} \subseteq \{z \in \mathbb{R}^n : \nabla g_j(x)^{\mathrm{T}}(z - x) > 0\}.$$

(ii) Show that if the Slater constraint qualification holds (see Lecture Notes 12) then the Mangasarian-Fromowitz constraint qualification holds (see Problem Set 5) at every feasible point of (CP). Hint: use part (i).

(iii) Show that if the MFCQ holds at at least one feasible point, then the SCQ holds, and hence by part (i) the MFCQ holds at every feasible point.

***Problem 2.** Consider the problem

$$\begin{aligned} & \min x_1 + \frac{2}{x_2} \qquad (0.1) \\ \text{s.t.} \quad & -x_2 + \frac{1}{2} \leq 0 \\ & -x_1 + x_2^2 \leq 0 \\ & x_2 \geq 0. \end{aligned}$$

(i) Show that (0.1) is a convex programming problem.

(ii) Use the method of Lagrange multipliers convex optimality conditions to solve this problem.

***Problem 3.** Consider the problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & c^{\mathrm{T}} x - \mu \sum_i \ln x_i \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0, \end{aligned} \tag{0.2}$$

where we set $-\ln x_i = +\infty$ for $x_i \leq 0$, and where $\mu > 0$.

(i) Write down the Lagrangian and determine the regions where its values are finite, $+\infty$ and $-\infty$.
(ii) Show that the Lagrangian primal is equivalent to (0.2). (You should do this directly instead of referring to Theorem 2.1 of Lecture 12.)
(iii) Show that the Lagrangian dual is equivalent to

$$\begin{aligned} \max_{(y,s)} \ & b^{\mathrm{T}} y + \mu \sum_i \ln s_i \\ \text{s.t.} \quad & A^{\mathrm{T}} y + s = c \\ & s \geq 0. \end{aligned}$$

(iv) What is the Slater condition in this case?
(v) In the case where (P) satisfies the SCQ and has an optimal solution, argue that this optimal solution is unique and characterised by the existence of (y, s) such that the following system is satisfied:

$$\begin{aligned} Ax &= b \\ A^{\mathrm{T}} y + s &= c \\ Xe &= \mu S^{-1} e \\ x, s &> 0, \end{aligned}$$

where $X = \mathrm{Diag}(x)$ and $S = \mathrm{Diag}(s)$ are the diagonal matrices with x and s on their diagonals. Conclude that in this case (D) also satisfies the (SCQ).
(vi) Do the same for the dual.