1. (a) Briefly explain the essential difference between linesearch and trust-region methods for unconstrained minimization. [8 marks]

(b) Let $\mathbf{x} \in \mathbb{R}^n$. Write down first-order necessary optimality conditions for a point $\mathbf{x}_*$ to be a local minimizer of $f(\mathbf{x})$ subject to the vector of inequality constraints $\mathbf{c}(\mathbf{x}) \geq \mathbf{0}$. [2 marks]

(c) Consider the "trust-region" subproblem

$$\underset{\mathbf{x} \in \mathbb{R}^n}{\text{minimize}} \quad \mathbf{g}^T\mathbf{x} + \tfrac{1}{2}\mathbf{x}^T\mathbf{B}\mathbf{x} \quad \text{subject to} \quad \|\mathbf{x}\|_2 \leq \Delta$$

for some scalar $\Delta > 0$. By noting that the constraint is equivalent to $\tfrac{1}{2}\mathbf{x}^T\mathbf{x} \leq \tfrac{1}{2}\Delta^2$ and using your answer to part (b), or otherwise, show that the solution $\mathbf{x}_*$ to this trust-region subproblem necessarily satisfies

$$(\mathbf{B} + \lambda_*\mathbf{I})\mathbf{x}_* = -\mathbf{g},$$

where $\mathbf{I}$ is the identity matrix and the scalar $\lambda_* \geq 0$. Show in addition that either $\lambda_* = 0$ or $\|\mathbf{x}\|_2 = \Delta$. [5 marks]

(d) Hence or otherwise, find the solution to the trust-region subproblem with data

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \Delta = 5/12$$

[Hint: a root of the nonlinear equation

$$\frac{1}{(1+\lambda)^2} + \frac{1}{(2+\lambda)^2} = 25/144$$

is $\lambda = 2$.]. [5 marks]

2. Consider the problem

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad x_1 + \tfrac{1}{2}x_2^2 \quad \text{subject to} \quad x_1 \geq 0$$

(a) What is the minimizer of this problem? What is the value of its Lagrange multiplier? [5 marks]

(b) Write down the logarithmic barrier function $\Phi(\mathbf{x}, \mu)$ for the problem. What is the minimizer $x(\mu)$ of the barrier function as a function of the barrier parameter μ? What Lagrange multiplier estimate does this minimizer give? [5 marks]

(c) Compute the Hessian matrix of the logarithmic barrier function. What are its eigenvalues at the minimizer of the barrier function? How do these eigenvalues behave as the barrier parameter decreases to zero? Does this mean that the Newton equations cannot be solved accurately on a computer? [5 marks]

(d) Find the primal-dual step at $\mathbf{x}(\mu)$ when μ is reduced to $\bar{\mu}$. How good is this step as an approximation for the minimizer of $\Phi(x, \bar{\mu})$? [5 marks]