Notes for Part 5: SQP methods for equality constrained optimization

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5 Sketches of proofs for Part 5

5.1 Proof of Theorem 5.1

The SQP search direction s_k and its associated Lagrange multiplier estimates y_{k+1} satisfy

$$B_k s_k - A_k^T y_{k+1} = -g_k \tag{5.1}$$

and

$$A_k s_k = -c_k. \tag{5.2}$$

Premultiplying (5.1) by s_k and using (5.2) gives that

$$s_k^T g_k = -s_k^T B_k s_k + s_k^T A_k^T y_{k+1} = -s_k^T B_k s_k - c_k^T y_{k+1} \tag{5.3}$$

Likewise (5.2) gives

$$\frac{1}{\mu_k} s_k^T A_k^T c_k = -\frac{\|c_k\|_2^2}{\mu_k}. \tag{5.4}$$

Combining (5.3) and (5.4), and using the positive definiteness of B_k, the Cauchy-Schwarz inequality and the fact that $s_k \neq 0$ if x_k is not critical, yields

$$s_k^T \nabla_x \Phi(x_k) = s_k^T \left(g_k + \frac{1}{\mu_k} A_k^T c_k \right) = -s_k^T B_k s_k - c_k^T y_{k+1} - \frac{\|c_k\|_2^2}{\mu_k} < -\|c_k\|_2 \left(\frac{\|c_k\|_2}{\mu_k} - \|y_{k+1}\|_2 \right) \leq 0$$

because of the required bound on μ_k.

5.2 Proof of Theorem 5.2

The proof is slightly complicated as it uses the calculus of non-differentiable functions. See Theorem 14.3.1 in

R. Fletcher, "Practical Methods of Optimization", Wiley (1987, 2nd edition),

where the converse result that if x_* is an isolated local minimizer of $\Phi(x, \rho)$ for which $c(x_*) = 0$, then x_* solves the given nonlinear program so long as ρ is sufficiently large, is also given. Moreover, Fletcher showns (Theorem 14.3.2) that x_* cannot be a local minimizer of $\Phi(x, \rho)$ when $\rho < \|y_*\|_D$.

5.3 Proof of Theorem 5.3

For small steps α, Taylor's theorem applied separately to f and c, along with (5.2), gives that

$$\begin{aligned}
\Phi(x_k + \alpha s_k, \rho_k) - \Phi(x_k, \rho_k) = & \ \alpha s_k^T g_k + \rho_k \left(\|c_k + \alpha A_k s_k\| - \|c_k\| \right) + O(\alpha^2) \\
= & \ \alpha s_k^T g_k + \rho_k \left(\|(1-\alpha) c_k\| - \|c_k\| \right) + O(\alpha^2) \\
= & \ \alpha \left(s_k^T g_k - \rho_k \|c_k\| \right) + O\left(\alpha^2\right)
\end{aligned}$$

Combining this with (5.3), and once again using the positive definiteness of B_k, the Hölder inequality and the fact that $s_k \neq 0$ if x_k is not critical, yields

$$\begin{aligned}
\Phi(x_k + \alpha s_k, \rho_k) - \Phi(x_k, \rho_k) = & \ -\alpha \left(s_k^T B_k s_k + c_k^T y_{k+1} + \rho_k \|c_k\| \right) + O(\alpha^2) \\
< & \ -\alpha \left(-\|c_k\| \|y_{k+1}\|_D + \rho_k \|c_k\| \right) + O(\alpha^2) \\
= & \ -\alpha \|c_k\| \left(\rho_k - \|y_{k+1}\|_D \right) + O(\alpha^2) < 0
\end{aligned}$$

because of the required bound on ρ_k, for sufficiently small α. Hence sufficiently small steps along s_k from non-critical x_k reduce $\Phi(x, \rho_k)$.