APPROXIMATE FACTORIZATION CONSTRAINT PRECONDITIONERS FOR SADDLE-POINT MATRICES*

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Abstract. We consider the application of the conjugate gradient method to the solution of large, symmetric indefinite linear systems. Special emphasis is put on the use of constraint preconditioners and a new factorization that can reduce the number of flops required by the preconditioning step. Results concerning the eigenvalues of the preconditioned matrix and its minimum polynomial are given. Numerical experiments validate these conclusions.

Key words. preconditioning, indefinite linear systems, Krylov subspace methods, conjugate gradient method

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1. Introduction. There are many areas where we wish to solve matrix systems of the form

$$(1.1) \quad \underbrace{\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_b = \underbrace{\begin{bmatrix} c \\ d \end{bmatrix}}_b,$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric and $B \in \mathbb{R}^{m \times n}$. We shall assume that $0 < m \leq n$, B is of full rank, and A is positive definite in the nullspace of B . The symmetric matrix $\mathcal{A}$ is indefinite (i.e., it has both positive and negative eigenvalues.) Such systems may arise in convex quadratic programming problems [13] and saddle-point systems (see [2] and [10, Chapters 7, 9]), as well as many other areas.

Solution of systems of equations of the form (1.1) can be achieved by a number of methods. For large, sparse or structured matrices iterative methods are an attractive option. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \text{span}\{b, \mathcal{A}b, \mathcal{A}^2b, \dots, \mathcal{A}^{n-1}b, \dots\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES), and generalized minimal residual method (GMRES) are all common iterative Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices, and GMRES for unsymmetric matrices [12].

MINRES and GMRES will find the solution of the linear system (1.1) within $n + m$ iterations in exact arithmetic, but CG may fail because of the indefiniteness of the system. GMRES is just an expensive way to implement MINRES when $\mathcal{A}$ is symmetric as here. For very large systems this upper bound on the number of

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iterations is not helpful and such methods would generally not be employed unless many fewer iterations were required in practice; it is often advantageous to use a preconditioner $\mathcal{P}$ with such iterative methods. The preconditioner should reduce the number of iterations required for convergence but not significantly increase the amount of computation required at each iteration [12, Chapter 13].

Keller, Gould, and Wathen [8] investigated the use of a preconditioner of the form

$$(1.2) \quad \mathcal{P} = \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix},$$

where G approximates but is not the same as A . They were able to prove various results about the eigenvalues and eigenvectors for the preconditioned systems $\mathcal{P}^{-1}\mathcal{A}$, where $\mathcal{A}$ and $\mathcal{P}$ are defined in (1.1) and (1.2), respectively; $\mathcal{P}$ is called a *constraint preconditioner*.

The system (1.1) is symmetric and indefinite, implying that we are unable to apply the CG method directly in a robust way since it might fail [3, section 6.3]. Gould, Hribar, and Nocedal [7] show how the CG method can still be used when solving quadratic programming problems provided a preconditioner of the form (1.2) is employed. We extend their method to the general problem of solving systems of the form (1.1) where a constraint preconditioner of the form (1.2) is also used in sections 2 and 3. In section 4 we introduce a new factorization for the preconditioner $\mathcal{P}$. This previously unpublished factorization was recently developed by Wil Schilders (Tech. Univ. Eindhoven/Philips) [11]. We shall refer to this as the Schilders factorization. In section 5 we give numerical examples to demonstrate the effectiveness of the use of this method. A related but different factorization is the variable-reduction method [4, Chapter 5].

2. CG method for the reduced system. We wish to solve the system (1.1) to find $[x^{*T}y^{*T}]^T$. Let Z be an $n \times (n-m)$ matrix spanning the nullspace of B ; then $BZ = 0$. The columns of B^T together with the columns of Z span $\mathbb{R}^n$ and any solution x^* of linear equations $Bx = d$ can be written as

$$(2.1) \quad x^* = B^T x_B^* + Z x_Z^*.$$

Substituting this into (1.1) gives

$$(2.2) \quad \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} B^T x_B^* + Z x_Z^* \\ y^* \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}.$$

Let us split the matrix $\mathcal{A}$ into a block 3×3 structure where each corner block is of dimension $m \times m$. We can also expand the vector $[x^{*T}y^{*T}]^T$ into a matrix-vector product, $\mathcal{H}\rho^*$. Let $Z = [Z_1^T Z_2^T]^T$. Expression (2.2) then becomes

$$(2.3) \quad \begin{bmatrix} A_{1,1} & A_{1,2} & B_1^T \\ A_{2,1} & A_{2,2} & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} B_1^T & Z_1 & 0 \\ B_2^T & Z_2 & 0 \\ 0 & 0 & I \end{bmatrix}}_{\mathcal{H}} \underbrace{\begin{bmatrix} x_B^* \\ x_Z^* \\ y^* \end{bmatrix}}_{\rho^*} = \begin{bmatrix} c_1 \\ c_2 \\ d \end{bmatrix}.$$

To maintain symmetry of our system we premultiply (2.3) by $\mathcal{H}^T$. Multiplying the matrix expression $\mathcal{H}^T \mathcal{A} \mathcal{H}$ and simplifying, we obtain the linear system

$$(2.4) \quad \begin{bmatrix} BAB^T & BAZ & BB^T \\ Z^T AB^T & Z^T AZ & 0 \\ BB^T & 0 & 0 \end{bmatrix} \begin{bmatrix} x_B^* \\ x_Z^* \\ y^* \end{bmatrix} = \begin{bmatrix} Bc \\ Z^T c \\ d \end{bmatrix}.$$

We observe that an $m \times m$ system determines x_B^* :

$$(2.5) \quad BB^T x_B^* = d.$$

Since B is of full rank, BB^T is symmetric, positive definite. We could therefore solve this system using the Cholesky factorization of BB^T if the dimension of the system is small enough [5, p. 143]. From (2.4), having found x_B^* , we can find x_Z^* by solving

$$(2.6) \quad A_{ZZ} x_Z^* = -c_Z,$$

where

$$A_{ZZ} = Z^T A Z, \quad c_Z = Z^T (AB^T x_B^* - c).$$

The matrix A is symmetric, positive definite in the nullspace of B ; hence $Z^T A Z$ is symmetric, positive definite. Anticipating our technique, we can apply the CG method to compute an approximate solution to the system (2.6). Substituting this into (2.1) will give us an approximate solution for x^* . Using (2.4) with (2.1) we obtain a system that can be solved to give an approximate solution for y^* :

$$(2.7) \quad BB^T y^* = B(c - Ax^*).$$

If we used a Cholesky factorization to find x_B^* , then this same factorization could be employed to solve (2.7).

Let us consider the practical application of the CG method to the system (2.6). As we noted in section 1, the use of preconditioning can improve the rate of convergence of the CG iteration. Let us assume that a preconditioner W_{ZZ} is given, where W_{ZZ} is a symmetric, positive definite matrix of dimension $n - m$. Let us consider the class of preconditioners of the form $W_{ZZ} = Z^T G Z$, where G is a symmetric matrix such that $Z^T G Z$ is positive definite. The preconditioned CG method can then be applied to the $(n - m)$ -dimensional reduced system $A_{ZZ} x_Z^* = -c_Z$ (see [7, Algorithm 2.1]).

3. CG method for the full system. Explicit use of Algorithm 2.1 from [7] would require knowledge of Z and solution of systems involving W_{ZZ} . The algorithm may, however, be written to make explicit use of the constraint preconditioner without the need for Z at all. Let us define the scaled projection matrix

$$(3.1) \quad P = Z(Z^T G Z)^{-1} Z^T.$$

We will later see that P is independent of the choice of nullspace basis Z .

ALGORITHM 3.1 (preconditioned CG in expanded form). *Choose an initial point x satisfying $Bx = d$, compute $r = Ax - c$, $g = Pr$, and $p = -g$. Repeat the following steps, until a convergence test is satisfied:*

$$\begin{aligned} x &\leftarrow x + \alpha p, \quad \text{where } \alpha = r^T g / p^T A p, \\ r^+ &= r + \alpha A p, \\ g^+ &= P r^+, \\ p &\leftarrow -g^+ + \beta p, \quad \text{where } \beta = (r^+)^T g^+ / r^T g, \\ g &\leftarrow g^+ \quad \text{and} \quad r \leftarrow r^+. \end{aligned}$$

Note the definition of g^+ via the projection step $g^+ = P r^+$. Following the terminology of [7], the vector g^+ will be called the *preconditioned residual*. It is defined to

be in the nullspace of B . We now wish to be able to apply the projection operator $Z(Z^T G Z)^{-1} Z^T$ without a representation of the nullspace basis Z . If G is nonsingular, then P can be expressed as

$$(3.2) \quad P = G^{-1}(I - B^T(BG^{-1}B^T)^{-1}BG^{-1}).$$

We can find g^+ by solving the system

$$(3.3) \quad \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} g^+ \\ v^+ \end{bmatrix} = \begin{bmatrix} r^+ \\ 0 \end{bmatrix}$$

whenever $z^T G z \neq 0$ for all nonzero z for which $Bz = 0$ [4, section 5.4.1]. The idea in Algorithm 3.2 below is to replace the projection step with the solution of (3.3) to define the same g^+ : this is why a constraint preconditioner of the form (1.2) is needed. A preconditioner which is not of constraint form would not suffice.

Discrepancy in the magnitudes of g^+ and r^+ can cause numerical difficulties for which Gould, Hribar, and Nocedal [7] suggest using a residual update strategy that redefines r^+ so that its norm is closer to that of g^+ . This dramatically reduces the roundoff errors in the projection operation in practice.

ALGORITHM 3.2 (preconditioned CG with residual update (PCG)). *Apply Algorithm 3.1 with the following changes:*

Replace $g = Pr$ by solving

$$\begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} g \\ v \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix},$$

set $y = v$ and $r \leftarrow r - B^T y$ at the end of the initialization stage,
replace $g^+ = Pr^+$ by solving (3.3),
replace $r \leftarrow r^+$ by $r \leftarrow r^+ - B^T v^+$.

The key point is that Algorithm 3.2 does not require the computation of any nullspace basis but is a CG procedure for a symmetric, positive definite system which yields the solution of the indefinite system (1.1).

We would therefore like to be able to solve the systems of the form (3.3) efficiently. If we were to use Gaussian elimination to solve this system then, in general, the number of flops required would be $\mathcal{O}((n+m)^3)$ [5, p. 98]. Although this may be reduced depending on the sparsity, it could be impractical for large systems. In the following section we introduce a different factorization for matrices of the form $\mathcal{P}$ in (1.2). If $2m \leq n$, then we can use this factorization to solve (3.3) in $\mathcal{O}((n-m)^3)$ flops. If $2m > n$, then the number of flops required is $\mathcal{O}(m^3)$.

4. Schilders' factorization for preconditioning step. The preconditioning step of solving (3.3) in Algorithm 3.2 involves a constraint preconditioner of the same form of $\mathcal{P}$ given in (1.2). Let us split $\mathcal{P}$ into a block 3×3 structure as we did for the matrix $\mathcal{A}$ in (2.3). Suppose we choose matrices $L_1 \in \mathbb{R}^{m \times m}$ and $L_2 \in \mathbb{R}^{(n-m) \times (n-m)}$ such that L_2 is nonsingular and assume that B_1 is nonsingular; then we can factorize $\mathcal{P}$ in the following manner:

$$(4.1) \quad \mathcal{P} = \underbrace{\begin{bmatrix} B_1^T & 0 & L_1 \\ B_2^T & L_2 & E \\ 0 & 0 & I \end{bmatrix}}_{\mathcal{P}_1} \underbrace{\begin{bmatrix} D_1 & 0 & I \\ 0 & D_2 & 0 \\ I & 0 & 0 \end{bmatrix}}_{\mathcal{P}_2} \underbrace{\begin{bmatrix} B_1 & B_2 & 0 \\ 0 & L_2^T & 0 \\ L_1^T & E^T & I \end{bmatrix}}_{\mathcal{P}_3},$$

where

$$(4.2) \quad D_1 = B_1^{-T} G_{1,1} B_1^{-1} - L_1^T B_1^{-1} - B_1^{-T} L_1,$$

$$(4.3) \quad D_2 = L_2^{-1} (G_{2,2} - B_2^T D_1 B_2 - E B_2 - B_2^T E^T) L_2^{-T},$$

$$(4.4) \quad E = G_{2,1} B_1^{-1} - B_2^T D_1 - B_2^T L_1^T B_1^{-1}.$$

We should emphasize that ensuring that B_1 is nonsingular by permutation of columns, if necessary, is an important and nontrivial practical issue.

Equations (4.2), (4.3), and (4.4) are such that we can choose the matrix G and use this to define D_1 , D_2 , and E , or we could choose matrices D_1 , D_2 , and E that then define the matrix G . If we choose to use the second option, then we need to make sure that G satisfies several criteria set out in sections 2 and 3. These are

- G is nonsingular, and
- $Z^T G Z$ is symmetric, positive definite.

Since $\mathcal{P}_1 (= \mathcal{P}_3^T)$ is nonsingular, G will certainly satisfy these criteria if $\mathcal{P}_2$ is symmetric, positive definite. This is equivalent to the simple condition that D_2 is symmetric, positive definite.

In using the factorization (4.1) to solve (3.3), we take $L_2 = I$ for simplicity. We then need to calculate the LU factorization of $B_1 \in \mathbb{R}^{m \times m}$ and the Cholesky factorization of $D_2 \in \mathbb{R}^{(n-m) \times (n-m)}$ once and then use these factored forms in each iteration. Note that we need only choose D_2 to be symmetric, positive definite, so we can effectively choose how much work is involved in the factorization of D_2 : the choice $D_2 = I$ will give a less effective preconditioner but renders this Cholesky factorization trivial. If i iterations are carried out in Algorithm 3.2, then there are $i + 1$ solves carried out using the factorization (4.1). For general B , D_1 , D_2 , E , L_1 , and L_2 , the total number of flops required by PCG is given by

$$\text{no. flops.} \sim \frac{2}{3}(2(n-m)^3 + m^3) + 2(i+1)(13m^2 + 3n^2).$$

However, if D_2 and L_2 are diagonal, then the number of flops required reduces to

$$\text{no. flops.} \sim \frac{2}{3}m^3 + 4m(i+1)(5m + 3n).$$

We would like to balance this reduction in the number of flops with the possible increase in the number of iterations carried out.

Keller, Gould, and Wathen give a bound on the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$. Proof of the following theorem can be found in [8].

THEOREM 4.1. *Let $\mathcal{A} \in \mathbb{R}^{(n+m) \times (n+m)}$ be a symmetric and indefinite matrix of the form*

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix},$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric and $B \in \mathbb{R}^{m \times n}$ is of full rank. Assume Z is an $n \times (n-m)$ basis for the nullspace of B . Preconditioning $\mathcal{A}$ by a matrix of the form

$$\mathcal{P} = \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix},$$

where $G \in \mathbb{R}^{n \times n}$ is symmetric, $G \neq A$, and $B \in \mathbb{R}^{m \times n}$ is as above, implies that

- the matrix $\mathcal{P}^{-1}\mathcal{A}$ has
 1. an eigenvalue at 1 with multiplicity $2m$, and
 2. $n - m$ eigenvalues λ which are defined by the generalized eigenvalue problem $Z^T A Z x_z = \lambda Z^T G Z x_z$,
- the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most $n - m + 2$.

Let us consider what happens if $G_{1,1} = A_{1,1}$, $G_{1,2} = G_{2,1}^T = A_{1,2}$, and we choose some matrix D_2 which is symmetric, positive definite. Equation (4.3) can be rearranged to give an explicit expression for $G_{2,2}$: this is given in Theorem 4.2. We assume that $G_{2,2} \neq A_{2,2}$. By choosing D_2 and using the factorization (4.1) to solve (3.3), we will never have to explicitly form the matrix $\mathcal{P}$.

Theorem 4.1 reveals that the preconditioned system $\mathcal{P}^{-1}\mathcal{A}$ has an eigenvalue at 1 with multiplicity $2m$, and $n - m$ eigenvalues which are defined by the generalized eigenvalue problem $Z^T A Z x_z = \lambda Z^T G Z x_z$. Noting that B_1 is nonsingular, we observe that $Z = [-B_2^T B_1^{-T} \ I]^T L_2^{-T}$ is an $n \times (n - m)$ basis for the nullspace of B . Using the above values of $G_{1,1}$, $G_{1,2}$, $G_{2,1}$, and $G_{2,2}$, it is straightforward to show that the $n - m$ eigenvalues can be defined by

$$(4.5) \quad \widehat{D}_2 x_z = \lambda D_2 x_z,$$

where

$$(4.6) \quad \widehat{D}_2 = L_2^{-1}(A_{2,2} - B_2^T D_1 B_2 - E B_2 - B_2^T E^T) L_2^{-T}.$$

In the optimization community, $\widehat{D}_2$ is called the reduced Hessian.

Note that both D_2 and $\widehat{D}_2$ are symmetric, positive definite, so the eigenvalues are all real. Using the preconditioning matrix $\mathcal{P}$ with $G = [A_{1,1} \ A_{1,2}; A_{2,1} \ G_{2,2}]$, we can improve on the Krylov subspace dimension given in Theorem 4.1.

THEOREM 4.2. *Let $\mathcal{A} \in \mathbb{R}^{(n+m) \times (n+m)}$ be a symmetric and indefinite matrix of the form*

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix},$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric and $B \in \mathbb{R}^{m \times n}$ is of full rank with the first m columns linearly dependent of each other. Let $A = [A_{1,1} \ A_{1,2}; A_{2,1} \ A_{2,2}]$ and $B = [B_1 \ B_2]$, where $A_{1,1}, B_1 \in \mathbb{R}^{m \times m}$, $A_{1,2} \in \mathbb{R}^{m \times (n-m)}$, $A_{2,1}, B_2 \in \mathbb{R}^{(n-m) \times m}$, and $A_{2,2} \in \mathbb{R}^{(n-m) \times (n-m)}$.

Assume that $m < n$ and $\mathcal{A}$ is nonsingular. Let us choose any matrices $L_1 \in \mathbb{R}^{m \times m}$, $L_2, D_2 \in \mathbb{R}^{(n-m) \times (n-m)}$ such that L_2 is nonsingular and D_2 is symmetric, positive definite. Assume $\mathcal{A}$ is preconditioned by a matrix of the form

$$\mathcal{P} = \begin{bmatrix} A_{1,1} & A_{1,2} & B_1^T \\ A_{2,1} & G_{2,2} & B_2^T \\ B_1 & B_2 & 0 \end{bmatrix},$$

where

$$G_{2,2} = L_2 D_2 L_2^T - B_2^T B_1^{-T} A_{1,1} B_1^{-1} B_2 + A_{2,1} B_1^{-1} B_2 + B_2^T B_1^{-T} A_{1,2};$$

then the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most $n - m + 1$.

Proof. By assumption, B_1 is nonsingular so we can use Schilders' factorization (4.1) to express both $\mathcal{A}$ and $\mathcal{P}$. Using these factorizations we can express $\mathcal{P}^{-1}\mathcal{A}$ as

$$(4.7) \quad \mathcal{P}^{-1}\mathcal{A} = \begin{bmatrix} I & \Theta & 0 \\ 0 & L_2^{-T} D_2^{-1} \widehat{D}_2 L_2^T & 0 \\ 0 & \Upsilon & I \end{bmatrix},$$

where D_1 , $\widehat{D}_2$, and E are given by (4.2), (4.6), and (4.4), respectively, and

$$(4.8) \quad \Theta = B_1^{-1} B_2 (I - L_2^{-T} D_2^{-1} \widehat{D}_2 L_2^T),$$

$$(4.9) \quad \Upsilon = (E^T - L_1^T B_1^{-1} B_2) (I - L_2^{-T} D_2^{-1} \widehat{D}_2 L_2^T).$$

From the eigenvalues of the preconditioned system, it is evident that the characteristic polynomial of the preconditioned system is of the form

$$(\mathcal{P}^{-1} \mathcal{A} - I)^{2m} \prod_{i=1}^{n-m} (\mathcal{P}^{-1} \mathcal{A} - \lambda_i I).$$

To prove the upper bound on the dimension of the Krylov subspace we need to show that the degree of the minimum polynomial is less than or equal to $n - m + 1$. Expanding the polynomial $\prod_{i=1}^{n-m} (\mathcal{P}^{-1} \mathcal{A} - \lambda_i I)$ of degree $n - m$, we obtain a matrix of the form

$$(4.10) \quad \begin{bmatrix} \prod_{i=1}^{n-m} (1 - \lambda_i) I & f_{n-m}(\Theta) & 0 \\ 0 & \prod_{i=1}^{n-m} (S - \lambda_i I) & 0 \\ 0 & f_{n-m}(\Upsilon) & \prod_{i=1}^{n-m} (1 - \lambda_i) I \end{bmatrix},$$

where $S = L_2^{-T} D_2^{-1} \widehat{D}_2 L_2^T$ and $f_{n-m}(\cdot)$ is defined by the recursive formula

$$f_{n-m}(\Gamma) = \Gamma \prod_{i=1}^{n-m-1} (1 - \lambda_i) I + f_{n-m-1}(\Gamma) (S - \lambda_{n-m} I)$$

with base case $f_1(\Gamma) = \Gamma$. Now let us premultiply the matrix (4.10) by $\mathcal{P}^{-1} \mathcal{A} - I$ to give

$$(4.11) \quad \begin{bmatrix} 0 & \Theta \prod_{i=1}^{n-m} (S - \lambda_i I) & 0 \\ 0 & (S - I) \prod_{i=1}^{n-m} (S - \lambda_i I) & 0 \\ 0 & \Upsilon \prod_{i=1}^{n-m} (S - \lambda_i I) & 0 \end{bmatrix}.$$

We note that the $(1, 2)$, $(2, 2)$, and $(3, 2)$ entries are in fact zero, since the λ_i ($i = 1, \dots, n - m$) are eigenvalues of S , which is similar to a symmetric matrix and thus diagonalizable. Hence, the degree of the minimum polynomial of $\mathcal{P}^{-1} \mathcal{A}$ is less than or equal to $n - m + 1$. $\square$

If at least one of the eigenvalues defined by (4.5) has multiplicity greater than one, then we can sharpen this bound.

THEOREM 4.3. *Suppose we precondition our system $\mathcal{A}$ by a matrix $\mathcal{P}$ as defined in Theorem 4.2. Assume that there exists $\alpha \in \mathbb{R}$ and $1 \leq j < n - m$ such that the eigenvalues defined by (4.5) can be “ordered” to satisfy*

$$\begin{aligned} \lambda_i &\neq \alpha, & i = 1, \dots, j, \\ \lambda_i &= \alpha & i = j + 1, \dots, n - m; \end{aligned}$$

then the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1} \mathcal{A}, b)$ is at most $j + 2$.

Proof. Following the proof of Theorem 4.2 we can show that

$$(4.12) \quad \prod_{i=1}^{j+1} (\mathcal{P}^{-1} \mathcal{A} - \lambda_i I) = \begin{bmatrix} \prod_{i=1}^{j+1} (1 - \lambda_i) I & f_{j+1}(\Theta) & 0 \\ 0 & \prod_{i=1}^{j+1} (S - \lambda_i I) & 0 \\ 0 & f_{j+1}(\Upsilon) & \prod_{i=1}^{j+1} (1 - \lambda_i) I \end{bmatrix},$$

where Θ , Υ , S , and $f_{j+1}(\cdot)$ are as defined previously. Now let us premultiply the matrix (4.12) by $\mathcal{P}^{-1}\mathcal{A} - I$ to give

$$(4.13) \quad \begin{bmatrix} 0 & \Theta \prod_{i=1}^{j+1} (S - \lambda_i I) & 0 \\ 0 & (S - I) \prod_{i=1}^{j+1} (S - \lambda_i I) & 0 \\ 0 & \Upsilon \prod_{i=1}^{j+1} (S - \lambda_i I) & 0 \end{bmatrix}.$$

As in the proof of Theorem 4.2, we note that the $(1, 2)$, $(2, 2)$, and $(3, 2)$ entries are in fact zero. Hence, the degree of the minimum polynomial of $\mathcal{P}^{-1}\mathcal{A}$ is less than or equal to $j + 2$. $\square$

THEOREM 4.4. *Suppose we precondition our system $\mathcal{A}$ by a matrix $\mathcal{P}$ as defined in Theorem 4.2. Assume that the matrix $A_{2,2}$ is symmetric, positive definite. Let us define $D_2 = L_2^{-1}A_{2,2}L_2^{-T}$; then the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ has dimension at most $\min(2m + 2, n - m + 1)$.*

Proof. Let us expand the difference $\widehat{D}_2 - D_2$:

$$\begin{aligned} \widehat{D}_2 - D_2 &= L_2^{-1}(A_{2,2} + B_1^T B_1^{-T} A_{1,1} B_1^{-1} B_2 - A_{2,1} B_1^{-1} B_2 - B_2^T B_1^{-T} A_{1,2}) L_2^{-T} \\ &\quad - L_2^{-1} A_{2,2} L_2^{-T} \\ &= L_2^{-1}(B_2^T B_1^{-T} (A_{1,1} B_1^{-1} B_2 - A_{1,2}) - A_{2,1} B_1^{-1} B_2) L_2^{-T}. \end{aligned}$$

Now,

$$\begin{aligned} \text{rank}(\widehat{D}_2 - D_2) &\leq \min[\min[\text{rank}(B_2), \min[\text{rank}(A_{1,1}), \text{rank}(B_2)]] + \text{rank}(A_{1,2})] \\ &\quad + \min[\text{rank}(A_{2,1}), \text{rank}(B_2)], n - m] \\ &\leq \min(2m, n - m). \end{aligned}$$

Since D_2 is, by assumption, positive definite, we may write $D_2 = WW^T$ for some nonsingular W . Thus

$$W^{-1}\widehat{D}_2 W^{-T} = I + W^{-1}L_2^{-1}(B_2^T B_1^{-T} (A_{1,1} B_1^{-1} B_2 - A_{1,2}) - A_{2,1} B_1^{-1} B_2) L_2^{-T} W^{-T}$$

differs from the identity matrix by a matrix of rank at most $\min(2m, n - m)$, and hence there are at most $\min(2m, n - m)$ nonunit eigenvalues defined by the generalized eigenvalue problem (4.5). We can therefore arrange the eigenvalues such that there exists $1 \leq j \leq \min(2m, n - m)$ and

$$\begin{aligned} \lambda_i &\neq 1, \quad i = 1, \dots, j, \\ \lambda_i &= 1, \quad i = j + 1, \dots, n - m. \end{aligned}$$

From Theorems 4.2 and 4.3 we conclude that the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most $\min(2m + 2, n - m + 1)$. $\square$

5. Numerical examples. We apply Algorithm 3.2 to solve a selection of problems from the CUTer collection [6], and also a set of Stokes saddle-point problems generated using the `ifiss` incompressible flow software associated with the book [2]. Any simple bounds in the CUTer problems are removed to create a problem of the form

$$\begin{aligned} \min_x f(x) &= \frac{1}{2}x^T A x + s^T x \\ &\text{subject to } Bx = t. \end{aligned}$$

For both sets of problems, the matrix $A \in \mathbb{R}^{n \times n}$ is symmetric and $B \in \mathbb{R}^{m \times n}$ is the full row rank matrix of linear constraints; A is positive definite in the nullspace

TABLE 5.1

Values of m , n , nonzeros in A , and nonzeros in B for the test problems.

Problem	m	n	$\text{nz}(A)$	$\text{nz}(B)$
CONT-050	2401	2597	2597	12005
CONT-101	10098	10197	2700	49599
CONT-300	90298	90597	90597	448799
CONT1-10	9801	10197	10197	49005
CVXQP1	5000	10000	69968	14998
CVXQP2	2500	10000	69968	7499
CVXQP3	7500	10000	69968	22497
DUAL1	1	85	7031	85
DUAL2	1	96	8920	96
DUAL3	1	111	12105	111
DUAL4	1	75	5523	75
KSIP	1021	1001	20	21002
MOSARQP1	3200	700	2590	4122
PRIMALC1	9	239	229	2079
PRIMALC2	7	238	230	1624
PRIMALC5	8	295	286	2304
PRIMALC8	8	528	519	4168
STOKES1	288	2178	28418	10452
STOKES2	1088	8450	122206	44306
STOKES3	769	5890	82582	30483
STOKES4	2945	23042	345234	126799
STOKES5	2112	5890	82582	30064
STOKES6	8448	23042	345234	129124
STOKES7	255	2178	28568	3826
STOKES8	1023	8450	122506	16771

of B (in fact on the whole space for the Stokes problems), and vectors x , s , and t have appropriate dimensions. A permuted lu factorization of B^T is used to find a permutation matrix, M , such that the first m columns of BM are linearly independent [5, Chapter 3]. We then set $B \leftarrow BM$, $A \leftarrow M^T AM$, $s \leftarrow M^T s$, and $x \leftarrow M^T x$. The assumption about the nonsingularity of B_1 will now hold.

The dimensions of the problems and number of nonzeros in the associated matrices A and B are given in Table 5.1. Tables 5.2–5.6 show the results of using Algorithm 3.2 with different preconditioners and a low tolerance. The results of using the preconditioner $\mathcal{P} = [I \ B^T; B \ 0]$ are shown in Table 5.2. This preconditioner is of the general form considered in [8] (see also [9]). Another possible choice for G is to use the diagonal entries of A [1]. The results of using this preconditioner are shown in Table 5.3. In Tables 5.4 and 5.5 we use preconditioners of the form considered in Theorems 4.2 and 4.4, respectively. The final preconditioner that we try will generally reproduce less of the matrix A in G than the previous two do, but we hope that this will be compensated by the iterations being “cheaper” to apply. The results are given in Table 5.6.

We compare the number of iterations and the amount of CPU time taken when we use an lu factorization of the preconditioner $\mathcal{P}$, and when we use the Schilders’ factorization for $\mathcal{P}$. In both cases, we carry out the factorization once and then use the factored forms in each iteration. When producing Schilders’ factorization, we set $L_1 = 0$ and $L_2 = I$. At present, we have not carried out substantial research into how we should choose the values of L_1 and L_2 . In an attempt to maintain sparsity, an approximate minimum degree reordering strategy is applied before carrying out any lu factorizations. Threshold pivoting is also used during the lu factorization stages: the default threshold value in MATLAB is used.

TABLE 5.2

Solution statistics: Comparisons between iterative PCG solvers when using lu and Schilders' factorizations in the preconditioning step with $\text{tol} = 10^{-2}$, $G = I$.

Problem	LU				Schilders'				Digits
	Its	FTime	ITime	Total	Its	FTime	ITime	Total	
CONT-050	1	1.18	0.76	2.03	1	6.95	0.45	7.49	13
CONT-101	1	21.43	6.53	28.73	1	123.38	2.91	127.06	9
CONT-300	—	—	—	—	—	—	—	—	—
CONT1-10	1	20.59	6.42	27.80	1	128.72	2.89	132.41	13
CVXQP1	2	1.93	1.36	3.40	2	74.49	2.16	76.75	14
CVXQP2	2	0.04	0.11	0.17	2	0.10	0.13	0.24	15
CVXQP3	2	3.19	3.10	6.63	2	23.96	1.02	25.32	15
DUAL1	5	0.00	0.00	0.00	5	0.01	0.01	0.01	15
DUAL2	4	0.00	0.00	0.01	4	0.01	0.01	0.02	13
DUAL3	5	0.00	0.01	0.01	5	0.01	0.01	0.02	14
DUAL4	4	0.00	0.00	0.00	4	0.01	0.00	0.01	15
KSIP	17	0.34	0.91	2.11	18	0.03	0.87	1.63	5
MOSARQP1	4	0.03	0.06	0.11	4	0.08	0.07	0.17	15
PRIMALC1	2	0.00	0.01	0.01	2	0.08	0.02	0.10	12
PRIMALC2	1	0.00	0.00	0.01	1	0.07	0.02	0.09	14
PRIMALC5	0	0.00	2.12	2.13	0	0.16	22.07	22.23	15
PRIMALC8	2	0.01	7.76	7.77	2	0.61	153.51	154.13	14
STOKES1	7	0.05	0.07	0.13	7	15.04	3.68	18.73	14
STOKES2	8	0.27	0.34	0.69	8	932.91	116.33	1049.33	13
STOKES3	2	0.26	0.12	0.44	2	280.59	16.54	297.18	11
STOKES4	2	0.95	0.56	1.83	—	—	—	—	—
STOKES5	2	0.20	0.26	0.62	2	69.71	6.01	75.87	12
STOKES6	2	1.29	1.54	3.84	—	—	—	—	—
STOKES7	9	0.02	0.07	0.10	9	10.48	0.32	10.80	14
STOKES8	10	0.09	0.40	0.51	10	662.66	5.04	667.73	13

Problem	LU			Schilders'		
	$\text{nz}(L(\mathcal{P}))$	$\text{nz}(U(\mathcal{P}))$	Total	$\text{nz}(L(D_2))$	$\text{nz}(U(D_2))$	Total
CONT-050	630887	607170	1238057	19306	19306	155305
CONT-101	5263734	6407090	11670824	4950	4950	700946
CONT-300	—	—	—	—	—	—
CONT1-10	5195636	6075386	11271022	78606	78606	833857
CVXQP1	708449	801586	1510035	1229895	1241503	2519154
CVXQP2	22168	25467	47635	17038	17038	46791
CVXQP3	1682000	1267180	2949180	257477	259336	578439
DUAL1	171	171	342	3570	3570	7227
DUAL2	193	193	386	4560	4560	9218
DUAL3	223	223	446	6105	6105	12323
DUAL4	151	151	302	2775	2775	5627
KSIP	23214	23214	46428	210	210	23424
MOSARQP1	24109	26831	50940	32887	32887	71296
PRIMALC1	2363	4424	6787	26565	26565	55227
PRIMALC2	1890	3500	5390	26796	26796	55230
PRIMALC5	2635	4923	7558	41321	41328	84969
PRIMALC8	4732	8884	13616	135460	135460	275104
STOKES1	21079	21079	42158	1335628	1334125	2683050
STOKES2	105503	105503	211006	23447464	23461548	46971065
STOKES3	63029	63029	126058	10908746	10877987	21825155
STOKES4	326595	326595	653190	—	—	—
STOKES5	135816	136414	272230	4010012	4050659	8095603
STOKES6	746422	745095	1491517	—	—	—
STOKES7	9155	9154	18309	1036995	1033457	2074914
STOKES8	44741	44741	89482	20095193	20083780	40198321

TABLE 5.3

Solution statistics: Comparisons between iterative PCG solvers when using lu and Schilders' factorizations in the preconditioning step with $\text{tol} = 10^{-2}$, $G = \text{diag}(A)$.

Problem	LU				Schilders'				Digits
	Its	FTime	ITime	Total	Its	FTime	ITime	Total	
CONT-050	1	1.28	0.74	2.12	1	6.98	0.45	7.53	13
CONT-101	1	10.38	5.17	16.33	1	32.47	2.84	36.09	10
CONT-300	—	—	—	—	—	—	—	—	—
CONT1-10	1	18.70	6.28	25.78	1	128.01	2.92	131.71	13
CVXQP1	1	1.61	0.93	2.64	1	74.61	1.47	76.18	11
CVXQP2	1	0.04	0.08	0.14	1	0.10	0.10	0.22	15
CVXQP3	1	3.45	2.40	6.20	1	23.14	0.80	24.28	13
DUAL1	5	0.00	0.00	0.00	5	0.01	0.00	0.01	12
DUAL2	4	0.00	0.00	0.01	4	0.01	0.01	0.02	14
DUAL3	5	0.00	0.01	0.01	5	0.01	0.01	0.02	13
DUAL4	4	0.00	0.00	0.00	4	0.01	0.00	0.01	13
KSIP	1	0.34	0.73	1.92	1	0.01	0.73	1.46	15
MOSARQP1	2	0.02	0.03	0.07	2	0.02	0.04	0.08	15
PRIMALC1	0	0.00	4.52	4.52	0	0.00	1.15	1.16	15
PRIMALC2	0	0.00	4.23	4.24	0	0.00	1.00	1.01	15
PRIMALC5	0	0.00	6.87	6.88	0	0.00	1.63	1.63	15
PRIMALC8	0	0.01	28.54	28.55	0	0.00	5.15	5.15	15
STOKES1	5	0.05	0.05	0.12	5	14.79	3.75	18.56	14
STOKES2	5	0.27	0.25	0.61	5	926.89	77.75	1004.72	15
STOKES3	2	0.34	0.12	0.54	2	284.06	17.13	301.24	12
STOKES4	2	0.88	0.55	1.74	—	—	—	—	—
STOKES5	2	0.21	0.27	0.63	2	67.73	6.54	74.43	12
STOKES6	2	1.30	1.55	3.85	—	—	—	—	—
STOKES7	4	0.02	0.04	0.06	4	10.51	0.16	10.67	15
STOKES8	4	0.09	0.19	0.31	4	663.26	2.35	665.64	13

Problem	LU			Schilders'		
	$\text{nz}(L(\mathcal{P}))$	$\text{nz}(U(\mathcal{P}))$	Total	$\text{nz}(L(D_2))$	$\text{nz}(U(D_2))$	Total
CONT-050	620089	639618	1259707	19306	19306	155305
CONT-101	4255279	2928002	7183281	4950	4950	693449
CONT-300	—	—	—	—	—	—
CONT1-10	5135085	5655789	10790874	78606	78606	833857
CVXQP1	723131	577861	1300992	1228384	1232830	2508970
CVXQP2	21880	23198	45078	16921	16713	46349
CVXQP3	1901464	1069000	2970464	257803	259757	579186
DUAL1	171	171	342	3570	3570	7227
DUAL2	193	193	386	4560	4560	9218
DUAL3	223	223	446	6105	6105	12323
DUAL4	151	151	302	2775	2775	5627
KSIP	22022	22024	44046	210	210	22424
MOSARQP1	7289	7380	14669	5145	5145	15462
PRIMALC1	248	4387	4635	230	229	2547
PRIMALC2	245	3478	3723	231	230	2092
PRIMALC5	303	4894	5197	287	286	2885
PRIMALC8	536	8855	9391	520	519	5215
STOKES1	21079	21079	42158	1335797	1334073	2683167
STOKES2	105503	105503	211006	23447428	23461508	46970989
STOKES3	63029	63029	126058	10908667	10878627	21825716
STOKES4	326595	326595	653190	—	—	—
STOKES5	135821	136395	272216	3972246	4017853	8025031
STOKES6	747054	745028	1492082	—	—	—
STOKES7	9155	9154	18309	1039584	1035774	2079820
STOKES8	44741	44741	89482	20085901	20077317	40182566

TABLE 5.4

Solution statistics: Comparisons between iterative PCG solvers when using lu and Schilders' factorizations in the preconditioning step with $tol = 10^{-2}$, $G_{1,1} = A_{1,1}$, $G_{1,2} = A_{1,2}$, and $D_2 = \text{diag}(\hat{D}_2)$.

Problem	LU				Schilders'				Digits
	Its	FTime	ITime	Total	Its	FTime	ITime	Total	
CONT-050	1	9.63	0.66	10.39	1	6.98	0.44	7.51	12
CONT-101	1	43.41	4.52	48.69	1	32.57	2.87	36.22	12
CONT-300	—	—	—	—	—	—	—	—	—
CONT1-10	1	159.98	5.70	166.48	1	128.90	2.91	132.61	12
CVXQP1	12	743.04	245.24	988.39	11	33.01	1.29	34.40	11
CVXQP2	3	125.36	22.92	148.29	3	0.56	0.15	0.72	14
CVXQP3	5	644.13	105.32	749.79	5	24.08	1.07	25.50	6
DUAL1	6	0.01	0.01	0.01	6	0.00	0.01	0.01	8
DUAL2	5	0.02	0.01	0.03	5	0.01	0.01	0.01	8
DUAL3	6	0.03	0.02	0.04	6	0.01	0.01	0.02	8
DUAL4	4	0.01	0.00	0.02	4	0.01	0.00	0.01	9
KSIP	2	0.08	0.73	1.53	2	0.01	0.74	1.70	11
MOSARQP1	2	0.05	0.04	0.11	2	0.02	0.04	0.07	15
PRIMALC1	0	0.06	4.54	4.60	0	0.00	1.17	1.17	15
PRIMALC2	0	0.00	4.22	4.23	0	0.00	1.01	1.01	15
PRIMALC5	0	0.01	6.84	6.85	0	0.00	1.62	1.62	15
PRIMALC8	0	0.01	28.59	28.60	0	0.00	5.15	5.16	15
STOKES1	52	25.97	50.53	76.51	52	4.09	0.58	4.68	2
STOKES2	—	—	—	—	187	237.68	11.64	249.41	—
STOKES3	45	453.50	374.39	827.94	46	36.96	1.61	38.65	2
STOKES4	—	—	—	—	—	—	—	—	—
STOKES5	35	182.53	179.95	362.63	34	8.40	1.53	10.08	3
STOKES6	—	—	—	—	—	—	—	—	—
STOKES7	8	15.63	6.25	21.89	8	1.47	0.06	1.53	14
STOKES8	11	943.87	167.57	1111.47	11	27.20	0.36	27.58	13

Problem	LU			Schilders'		
	$\text{nz}(L(\mathcal{P}))$	$\text{nz}(U(\mathcal{P}))$	Total	$\text{nz}(L(D_2))$	$\text{nz}(U(D_2))$	Total
CONT-050	595768	649016	1244784	196	196	117085
CONT-101	3681821	2776506	6458327	99	99	683747
CONT-300	—	—	—	—	—	—
CONT1-10	4867901	5674469	10542370	396	396	677437
CVXQP1	25787124	39299255	65086379	5000	5000	91004
CVXQP2	8243264	10825817	19069081	7500	7500	43683
CVXQP3	24645462	44499396	69144858	2500	2500	113266
DUAL1	3604	3634	7238	84	84	337
DUAL2	4579	4544	9123	95	95	377
DUAL3	6140	6103	12243	110	110	441
DUAL4	2882	2805	5687	74	74	297
KSIP	22024	22024	44048	20	20	22044
MOSARQP1	19264	19276	38540	2500	2500	10181
PRIMALC1	248	4387	4635	230	229	2547
PRIMALC2	245	3478	3723	231	230	2092
PRIMALC5	303	4894	5197	287	286	2885
PRIMALC8	536	8855	9391	520	519	5215
STOKES1	1684985	2104142	3789127	1890	1890	19875
STOKES2	—	—	—	7362	7362	87160
STOKES3	13281099	16195569	29476668	5121	5121	55944
STOKES4	—	—	—	—	—	—
STOKES5	7621013	13040444	20661457	3778	3778	62434
STOKES6	—	—	—	—	—	—
STOKES7	1308824	1706168	3014992	1923	1923	11678
STOKES8	23764396	31623689	55388085	7427	7427	48126

TABLE 5.5

Solution statistics: Comparisons between iterative PCG solvers when using lu and Schilders' factorizations in the preconditioning step with $\text{tol} = 10^{-2}$, $G_{1,1} = A_{1,1}$, $G_{1,2} = G_{2,1}^T = A_{1,2}$, and $D_2 = A_{2,2}$.

Problem	LU				Schilders'				Digits
	Its	FTime	ITime	Total	Its	FTime	ITime	Total	
CONT-050	1	8.35	0.66	9.12	1	0.06	0.45	0.60	11
CONT-101	1	41.44	4.50	46.71	1	0.45	2.84	4.06	6
CONT-300	—	—	—	—	1	14.48	60.19	98.69	—
CONT1-10	1	147.41	5.66	153.84	1	0.44	2.89	4.11	12
CVXQP1	401	952.69	9531.04	10483.85	134	0.09	15.05	15.24	3
CVXQP2	0	226.36	10.12	236.50	1	1.80	1.50	3.32	0
CVXQP3	29	587.99	522.23	1110.56	18	0.08	2.77	3.19	1
DUAL1	1	0.01	0.00	0.01	1	0.00	0.00	0.01	11
DUAL2	1	0.02	0.00	0.02	1	0.01	0.00	0.01	12
DUAL3	2	0.02	0.01	0.03	2	0.01	0.00	0.01	9
DUAL4	2	0.01	0.00	0.01	2	0.00	0.00	0.01	10
KSIP	0	0.07	6.60	7.40	0	0.00	1.59	2.31	15
MOSARQP1	0	0.04	172.38	172.44	0	0.01	142.10	142.13	15
PRIMALC1	0	0.01	4.24	4.25	0	0.00	1.14	1.15	15
PRIMALC2	0	0.00	4.00	4.01	0	0.00	1.00	1.00	15
PRIMALC5	0	0.01	6.55	6.55	0	0.00	1.71	1.71	15
PRIMALC8	0	0.01	28.06	28.07	0	0.00	5.10	5.10	15
STOKES1	91	23.21	91.12	114.35	92	0.05	1.22	1.28	2
STOKES2	—	—	—	—	560	0.31	39.42	39.82	—
STOKES3	79	450.28	688.39	1138.72	75	0.17	3.17	3.39	3
STOKES4	—	—	—	—	234	1.02	48.17	49.47	—
STOKES5	70	174.76	364.39	539.31	63	0.13	3.03	3.31	3
STOKES6	—	—	—	—	136	0.79	28.75	30.59	—
STOKES7	14	14.35	11.17	25.53	14	0.05	0.14	0.20	11
STOKES8	28	834.47	434.78	1269.28	28	0.28	1.25	1.55	7

Problem	LU			Schilders'		
	$\text{nz}(L(P))$	$\text{nz}(U(P))$	Total	$\text{nz}(L(D_2))$	$\text{nz}(U(D_2))$	Total
CONT-050	595768	648968	1244736	196	196	117085
CONT-101	3690797	2775736	6466533	99	99	683747
CONT-300	—	—	—	299	299	10794780
CONT1-10	4867901	5800598	10668499	396	396	677437
CVXQP1	31909093	44477738	76386831	23310	23734	128048
CVXQP2	14908190	16628011	31536201	847610	572256	1448549
CVXQP3	24497857	41965765	66463622	4926	4926	118118
DUAL1	3739	3739	7478	3568	3568	7305
DUAL2	4747	4747	9494	4554	4554	9295
DUAL3	6326	6326	12652	6104	6104	12429
DUAL4	2924	2924	5848	2775	2775	5699
KSIP	22024	22023	44047	20	19	22043
MOSARQP1	9821	8304	18125	2536	2186	9903
PRIMALC1	248	4387	4635	230	229	2547
PRIMALC2	245	3478	3723	231	230	2092
PRIMALC5	303	4894	5197	287	286	2885
PRIMALC8	536	8855	9391	520	519	5215
STOKES1	1684987	2104662	3789649	28475	28475	73045
STOKES2	—	—	—	190490	190490	453416
STOKES3	13362756	16195611	29558367	99430	99430	244562
STOKES4	—	—	—	620644	620715	1439851
STOKES5	7617212	13057170	20674382	79728	79728	214334
STOKES6	—	—	—	439216	452489	1122908
STOKES7	1314702	1707861	3022563	29666	29666	67164
STOKES8	23715838	31936839	55652677	177137	177137	387546

TABLE 5.6

Solution statistics: Comparisons between iterative PCG solvers when using lu and Schilders' factorizations in the preconditioning step with $\text{tol} = 10^{-2}$, $G_{1,1} = 0$, $G_{1,2} = 0$, and $D_2 = A_{2,2}$.

Problem	LU				Schilders'				Digits
	Its	FTime	ITime	Total	Its	FTime	ITime	Total	
CONT-050	1	0.72	0.38	1.20	1	0.06	0.43	0.59	11
CONT-101	1	5.39	2.50	8.66	1	0.46	2.85	4.09	10
CONT-300	1	324.00	98.27	446.28	1	14.59	59.78	98.46	14
CONT1-10	1	8.24	3.00	12.05	1	0.45	2.84	4.07	10
CVXQP1	119	29.27	330.01	359.39	104	0.09	10.20	10.39	2
CVXQP2	3	7.73	4.53	12.28	0	1.78	0.99	2.79	0
CVXQP3	16	7.21	18.37	25.91	16	0.07	2.11	2.52	4
DUAL1	1	0.01	0.00	0.01	1	0.00	0.00	0.01	15
DUAL2	1	0.01	0.00	0.02	1	0.01	0.00	0.01	15
DUAL3	2	0.02	0.00	0.02	2	0.01	0.00	0.01	14
DUAL4	2	0.01	0.00	0.01	2	0.00	0.00	0.01	12
KSIP	0	0.34	6.67	7.74	0	0.00	1.59	2.32	15
MOSARQP1	0	0.03	149.10	149.14	0	0.01	136.80	136.82	15
PRIMALC1	0	0.01	4.35	4.36	0	0.00	1.15	1.15	15
PRIMALC2	0	0.00	4.03	4.03	0	0.00	1.01	1.01	15
PRIMALC5	0	0.01	6.85	6.85	0	0.00	1.62	1.62	15
PRIMALC8	0	0.01	28.28	28.29	0	0.00	5.18	5.18	15
STOKES1	75	0.21	3.05	3.28	79	0.05	1.02	1.08	2
STOKES2	240	4.75	259.71	264.55	254	0.30	16.79	17.18	2
STOKES3	65	0.81	8.52	9.38	65	0.16	2.59	2.80	3
STOKES4	179	9.79	211.73	221.81	163	0.98	31.93	33.21	5
STOKES5	58	1.48	20.53	22.15	59	0.12	2.51	2.78	4
STOKES6	133	45.24	543.96	590.21	128	0.69	23.81	25.49	2
STOKES7	14	0.10	0.31	0.41	14	0.05	0.13	0.18	14
STOKES8	28	0.70	4.05	4.77	28	0.27	1.15	1.45	10

Problem	LU			Schilders'		
	$\text{nz}(L(\mathcal{P}))$	$\text{nz}(U(\mathcal{P}))$	Total	$\text{nz}(L(D_2))$	$\text{nz}(U(D_2))$	Total
CONT-050	293011	411916	704927	196	196	114684
CONT-101	1770989	2230310	4001299	99	99	681146
CONT-300	45690125	62556439	108246564	299	299	10704482
CONT1-10	2194946	3280161	5475107	396	396	667636
CVXQP1	4346056	3841931	8187987	23310	23734	89800
CVXQP2	1856579	1343030	3199609	847610	572256	1430081
CVXQP3	1831354	1842046	3673400	4926	4926	63978
DUAL1	3570	3738	7308	3568	3568	7222
DUAL2	4556	4746	9302	4554	4554	9205
DUAL3	6106	6326	12432	6104	6104	12320
DUAL4	2777	2925	5702	2775	2775	5626
KSIP	22022	22023	44045	20	19	22042
MOSARQP1	7325	7066	14391	2536	2186	9544
PRIMALC1	248	4387	4635	230	229	2547
PRIMALC2	245	3478	3723	231	230	2092
PRIMALC5	303	4894	5197	287	286	2885
PRIMALC8	536	8855	9391	520	519	5215
STOKES1	83694	181989	265683	28475	28475	69959
STOKES2	2329914	1548050	3877964	190490	190490	441945
STOKES3	258848	797838	1056686	99430	99430	236513
STOKES4	2486613	6207846	8694459	620644	620715	1409233
STOKES5	768367	542274	1310641	79728	79728	192276
STOKES6	8697371	6909772	15607143	439216	452489	1032170
STOKES7	50998	58625	109623	29666	29666	63539
STOKES8	340761	379503	720264	177137	177137	372599

TABLE 5.7

Solution statistics: Comparison of different preconditioners with the preferable factorization method and $\text{tol} = 10^{-6}$. (a) $G = I$, (b) $G = \text{diag}(A)$, (c) $G_{1,1} = A_{1,1}$, $G_{1,2} = G_{2,1}^T = A_{1,2}$, and $D_2 = A_{2,2}$, (d) $G_{1,1} = 0$, $G_{1,2} = G_{2,1}^T = 0$, and $D_2 = A_{2,2}$.

Name	LU (a)		LU (b)		Schilders' (c)		Schilders' (d)	
	Its	Time	Its	Time	Its	Time	Its	Time
CONT-050	3	2.68	1	2.06	4	1.13	4	1.09
CONT-101	7	45.98	1	16.21	2	5.14	2	5.11
CONT-300	—	—	—	—	4	155.35	4	153.56
CONT1-10	1	27.66	1	26.17	4	7.36	4	7.12
CVXQP1	16	9.04	9	5.98	773	85.96	601	58.43
CVXQP2	18	0.72	9	0.41	1	3.44	0	2.81
CVXQP3	12	15.93	6	11.35	161	21.60	141	15.79
DUAL1	38	0.02	37	0.02	3	0.01	3	0.01
DUAL2	22	0.02	21	0.02	4	0.02	3	0.02
DUAL3	24	0.03	24	0.03	2	0.02	2	0.02
DUAL4	14	0.01	13	0.01	3	0.01	3	0.01
KSIP	20	2.13	1	1.93	0	2.33	0	2.28
MOSARQP1	21	0.28	5	0.08	0	142.22	0	136.41
PRIMALC1	6	1.34	0	4.36	0	1.16	0	1.14
PRIMALC2	2	1.27	0	4.02	0	1.01	0	1.00
PRIMALC5	0	2.13	0	6.59	0	1.64	0	1.64
PRIMALC8	2	7.88	0	28.52	0	5.17	0	5.10
STOKES1	36	0.32	32	0.30	298	4.08	257	3.21
STOKES2	67	2.50	57	2.21	3275	230.97	1575	104.44
STOKES3	23	0.90	21	0.84	397	16.66	330	13.08
STOKES4	38	5.06	34	4.66	2093	425.20	1460	283.37
STOKES5	23	1.45	21	1.36	333	15.56	309	12.42
STOKES6	37	10.90	35	10.38	1382	281.50	1261	226.31
STOKES7	35	0.28	23	0.20	45	0.49	45	0.44
STOKES8	77	2.91	38	1.49	116	5.33	115	4.73

The number of iterations carried out is given in the column headed “Its.” We also measure the CPU time taken in carrying out the factorization of the preconditioners (FTime) and the CPU time then taken by Algorithm 3.2 when using these factored forms (ITime). The time spent finding an initial x which satisfies $Bx = d$ is excluded from the value given in Itime. The total CPU time used to solve the problem is given by “Total.” Each problem is solved ten times and the mean of the CPU times recorded. The minimum values of $\frac{1}{2}x^T Ax + s^T x$ calculated are also compared for all the problems, although this is not immediately applicable to the Stokes problems. The number of digits of agreement in these values are given in “Digits.” We observe that for the majority of the CUTer problems, the differences in rounding errors of the two factorization methods do not greatly affect the calculated minimum values. We note that for some of the Stokes problems at least one of the solutions is rather inaccurate, though we have no easy way of finding out which! An important factor in how well a certain factorization will perform is its sparsity; in particular, the number of nonzeros in the lu factors of $\mathcal{P}$, and the number of nonzeros in the factors of D_2 . These values are also given for each of our preconditioners, and the total amount of storage required for the various blocks required. Note that the matrices D_1 and E should not be generated when using Schilders’ factorization.

The diagonal form of D_2 used in the examples of Table 5.4 produces a significant

FIG. 5.1. Convergence of relative residuals $r^T g$ with the different preconditioning strategies from Table 5.7 for the problem DUAL1.

drop in the CPU times when we use the Schilders' factorization instead of the lu factorization of $\mathcal{P}$. However, this choice of D_2 is not so practical because we have to form the matrix $\hat{D}_2$, as defined in (4.6), and then use the diagonal. If $A_{2,2}$ is positive definite and sparse enough for us to consider factorizing it, then we can set $D_2 = A_{2,2}$ with $G_{1,1} = A_{1,1}$ and $G_{1,2} = G_{2,1}^T = A_{1,2}$. From Theorem 4.4 we know that the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most $\min(n - m + 1, 2m + 2)$. The results of using such a preconditioner are shown in Table 5.5. In particular, note the four DUAL* problems. For each of these $m = 1$, so the upper bound on the dimension of the Krylov subspace $\mathcal{K}(\mathcal{P}^{-1}\mathcal{A}, b)$ is at most 4. Our results support this claim.

All tests were performed on a dual Intel Xeon 3.20GHz machine with hyper-threading and 2GB of RAM. It was running Fedora Core 2 (Linux kernel 2.6.8) and all programs were written in MATLAB 7.0. We terminate the iteration when the value of $r^T g$ is reduced by at least a factor of 10^{-2} . If no result is given for a problem, then the machine we used ran out of memory during the factorization stage. This is indicated by— in the results tables. Consider the test problem CONT-300. We were unable to solve it by choosing G explicitly, as in Tables 5.2 and 5.3, and then factoring it because of the machine running out of memory. However, when we used Schilders' factorization with $D_2 = A_{2,2}$, our algorithm successfully terminated, as in Tables 5.5 and 5.6. We were also able to factor $\mathcal{P}$ with an lu factorization for $G_{1,1} = 0$, $G_{1,2} = G_{2,1}^T = 0$, and $D_2 = A_{2,2}$, but the factorization time was over 300 times larger than using Schilders' factorization, resulting in the total CPU time being at least 38 times larger.

It is clear that, for the majority of the problems, factoring $\mathcal{P}$ with an lu factorization is preferable when we choose G as in Tables 5.2 and 5.3. However, it is generally preferable to use Schilders' factorization for the other choices of preconditioner, as we would expect. Table 5.7 shows results for the factorization of preference with four of the preconditioners that we have considered. The iteration process is terminated

when $r^T g$ has been reduced by a factor of 10^{-6} . The number of iterations required and the total CPU time is recorded. We observe that Schilders' factorization with $G_{1,1} = 0$, $G_{1,2} = 0$, and $D_2 = A_{2,2}$ is preferable, or of equal preference, for twelve out of the seventeen CUTER test problems. However, setting $G = \text{diag}(A)$ and then carrying out an `lu` factorization is better for the Stokes problems tested because the number of iterations required is greatly reduced. The convergence of the relative residuals of $r^T g$ are plotted in Figure 5.1 for the problem DUAL1. The line labeled "(c)" corresponds to a preconditioner which satisfies the assumptions of Theorem 4.4. The plot supports this theorem.

6. Conclusion. In this paper, we have discussed the use of the PCG method for solving indefinite linear systems. We have investigated the use of a new factorization for constraint preconditioners and shown how this can be used in incomplete forms to decrease the number of flops required overall by the preconditioned iterative method. We obtained an upper bound on the number of iterations required to solve systems of the form (1.1) by means of appropriate Krylov subspace methods by using a minimum polynomial argument.

We have provided computational evidence that the PCG method works well for nontrivial quadratic programming problems; however, it would not be the method of choice for the Stokes saddle-point problems tested. Indeed, problem-based preconditioned iterative methods which have optimal computational complexity exist for these problems (see [2, Chapter 6]).

In some cases, the use of Schilders' factorization to define a preconditioner can speed up the solution time by a factor of 5 compared to using the `lu` factorization of a preconditioner for which G has been chosen. For larger dimensional problems, we have shown an example where Schilders' factorization approach enables fast solution but the `lu` factorization with popular choices of preconditioner fails because of memory costs.

The cost of finding m linearly independent columns in the matrix B via a permuted `lu` factorization is often higher than we would like. We plan to investigate different strategies for this in the future, and how these may change the number of iterations required to solve the problems.

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