Spectral accuracy and conformal maps

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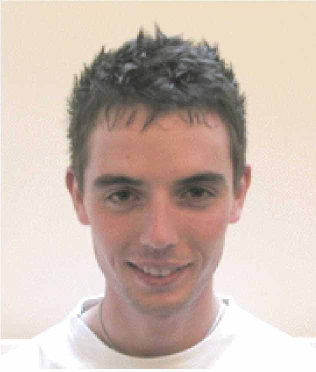
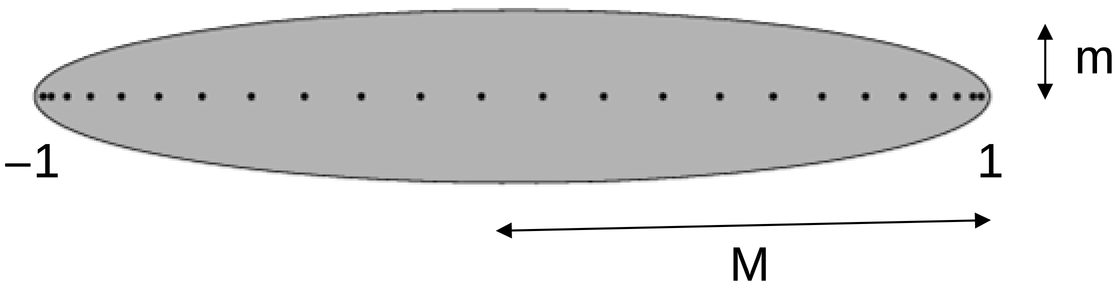


Nick Hale

E_r = "r-ellipse" with foci ±1, semimajor + semiminor axes = $r > 1$.



$r = M+m$

Suppose f is analytic in E_r with $|f(z)| \le K$.

THEOREM. For degree n polynomials, i.e. n+1 points:

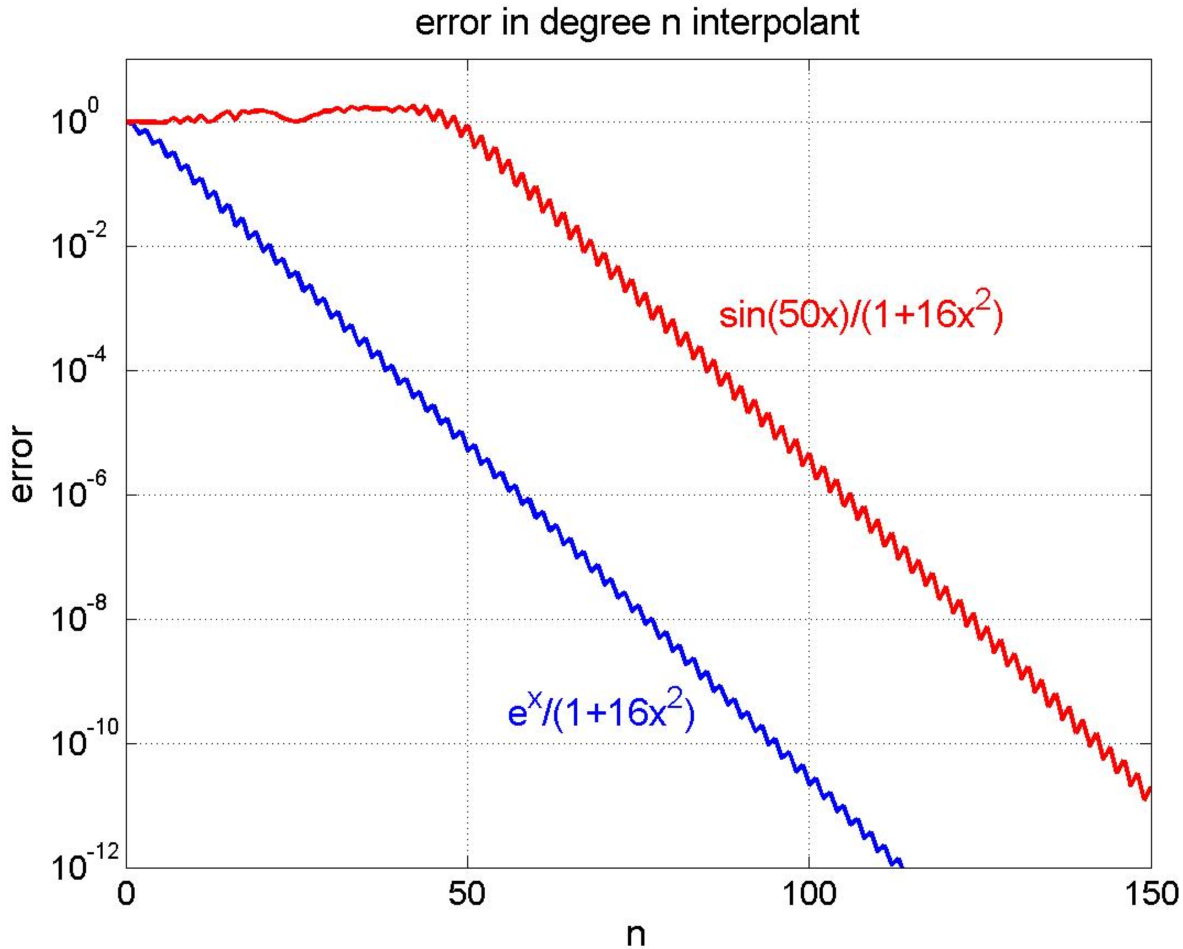
Interpolation in Chebyshev pts: $\|f - p_n\| \le 4K\, r^{-n}/(r-1)$ APPROXIMATION

Gauss quadrature: $|I - I_n| \le 5K\, r^{-2n}/(r^2-1)$ QUADRATURE

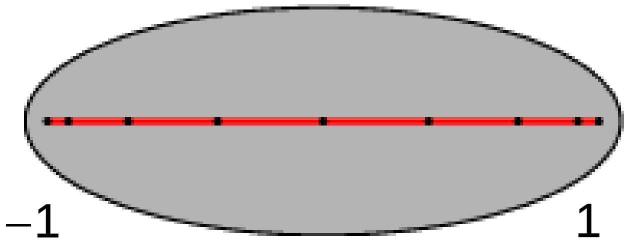
$O(r^{-n})$ accuracy for derivatives too. SPECTRAL METHODS

The essence of such results dates to de la Vallée Poussin & Bernstein, 1910s. Finding refs for these precise bounds, however, is hard.

EXAMPLE



error in degree n interpolant
error
n
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
0
50
100
150
$\sin(50x)/(1+16x^2)$
$e^x/(1+16x^2)$



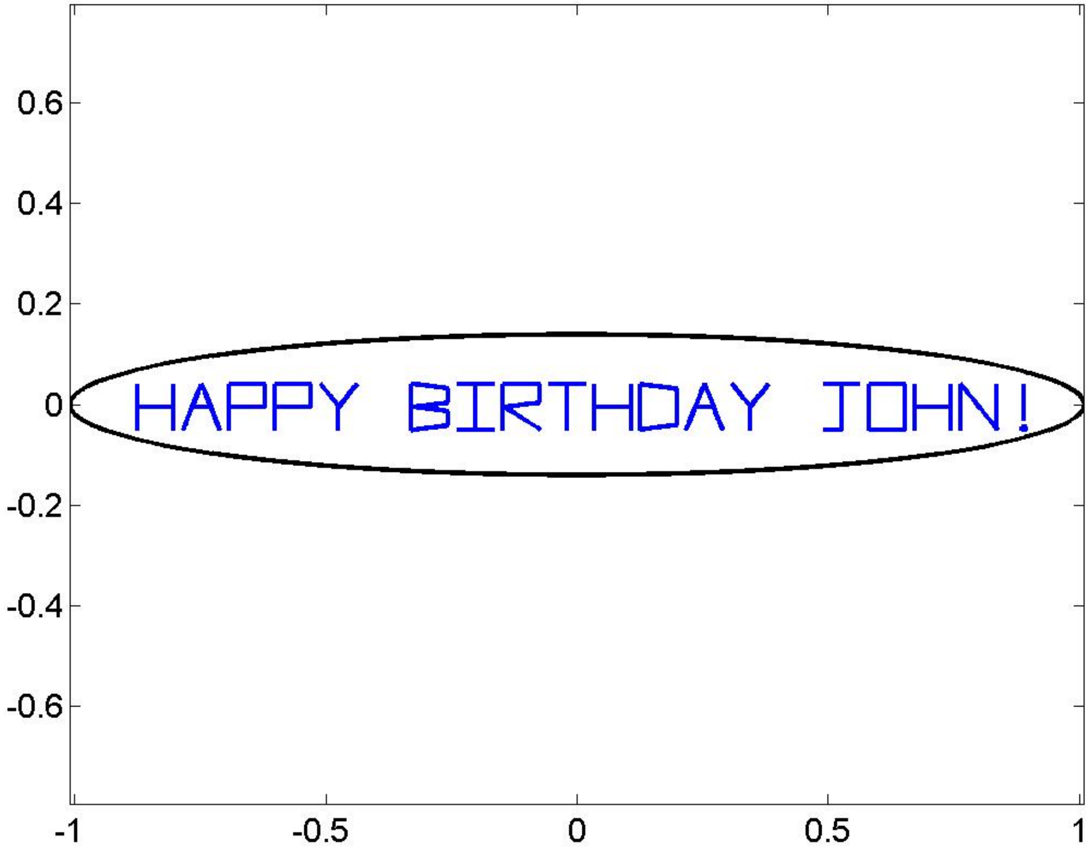
Q: Where do r-ellipses come from?
A: From using polynomials to approximate.

← potential theory

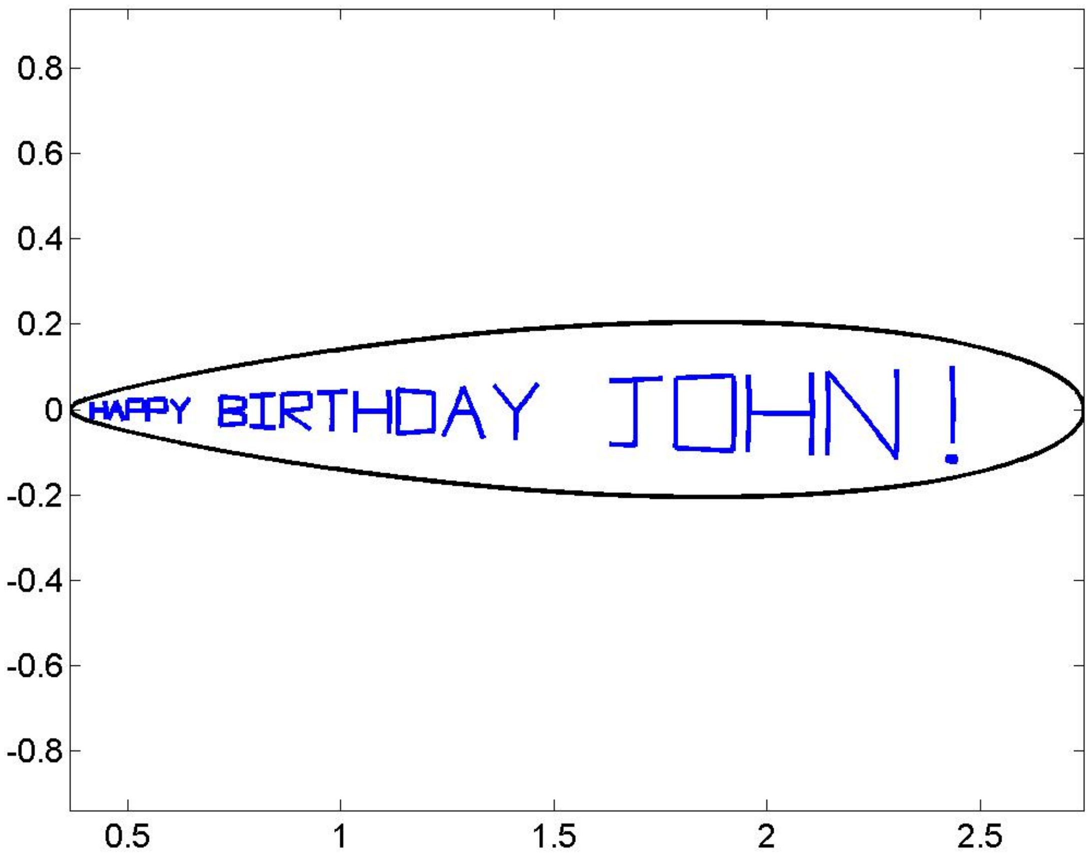
Q: Why do we have to use polynomials?
A: We don't!

By introducing a conformal map, we can change the approximations and change their convergence behaviour.

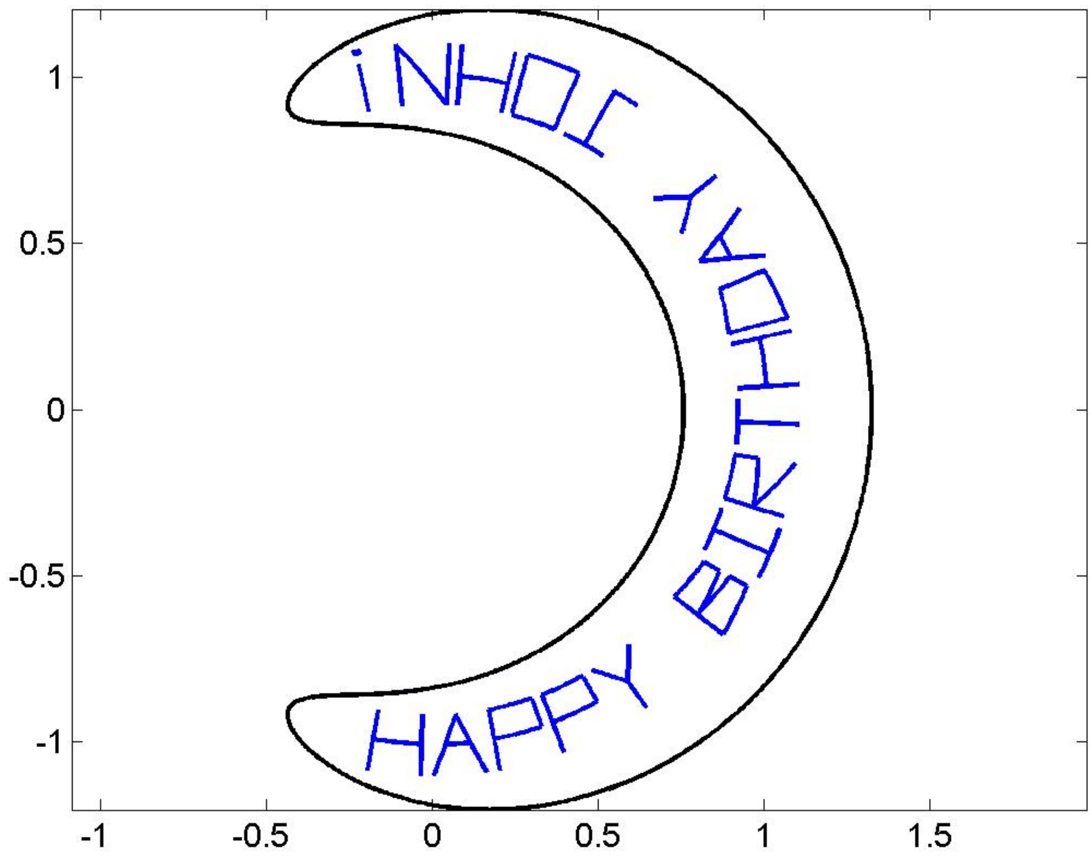
For example, here is an r-ellipse in the s-plane $(r=1.15)$.



Here is the image under the mapping z = exp(s).

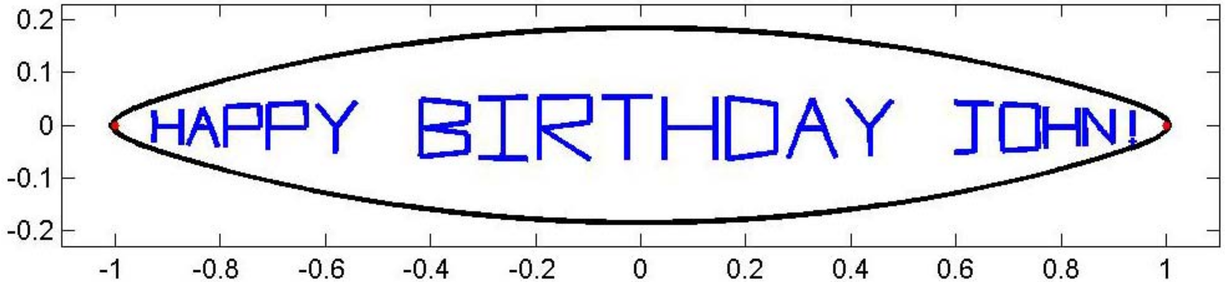


Here is the image under z = exp(2is).

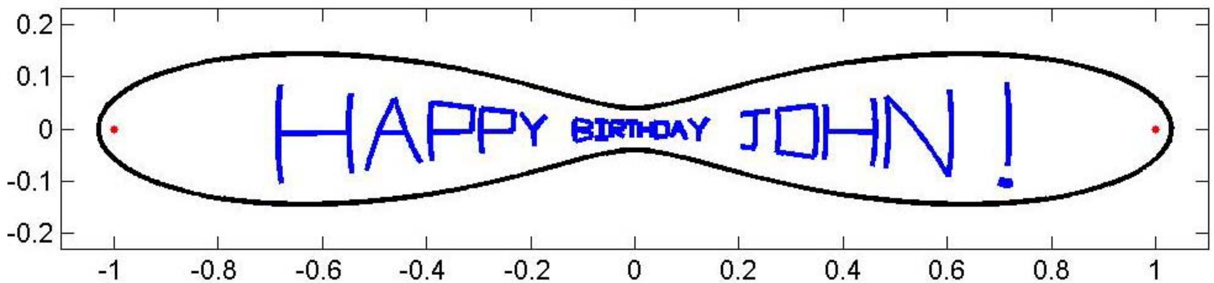


Actually, we use maps g(s) such that g(±1) = ±1, g([−1,1]) = [−1,1].

g(s) = tanh(s)/tanh(1):

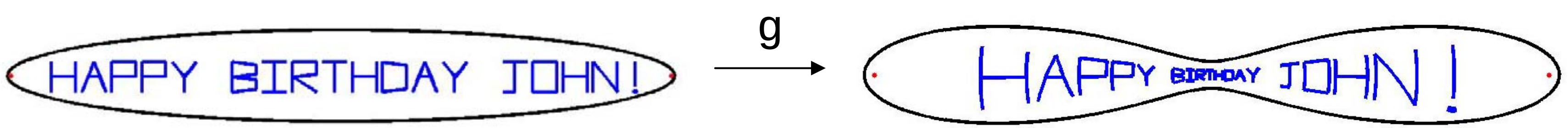


g(s) = sinh(3s)/sinh(3):



This gives transplanted approximation or transplanted quadrature, with $O(r^{-n})$ or $O(r^{-2n})$ convergence if f is analytic in the region shown.

For example,



Gauss quadrature here...

...gives us a non-polynomial **transplanted quadrature rule** here, which will be accurate for functions analytic in this region

Transplanted integral: $$\int_{-1}^{1} f(x)\, dx = \int_{-1}^{1} f(g(s))\, g'(s)\, ds$$

Transplanted qudarature: $$\int_{-1}^{1} f(x)\, dx \approx \sum_{k} w_k\, g'(x_k)\, f(x_k)$$

Suppose f is analytic in $g(E_r)$ with $|f(z)| \le K$.

THEOREM. For approximation by transformed polynomials, $\|f(x) - p_n(g^{-1}(x))\| \le 4K\, r^{-n}/(r-1)$

TRANSPLANTED APPROXIMATION

Define in addition $G = \max|g'(s)|$, $s \in E_r$.

THEOREM. For transplanted Gauss quadrature,

$$|I - I_n| \le 5GK\, r^{-2n}/(r^2-1)$$

TRANSPLANTED QUADRATURE

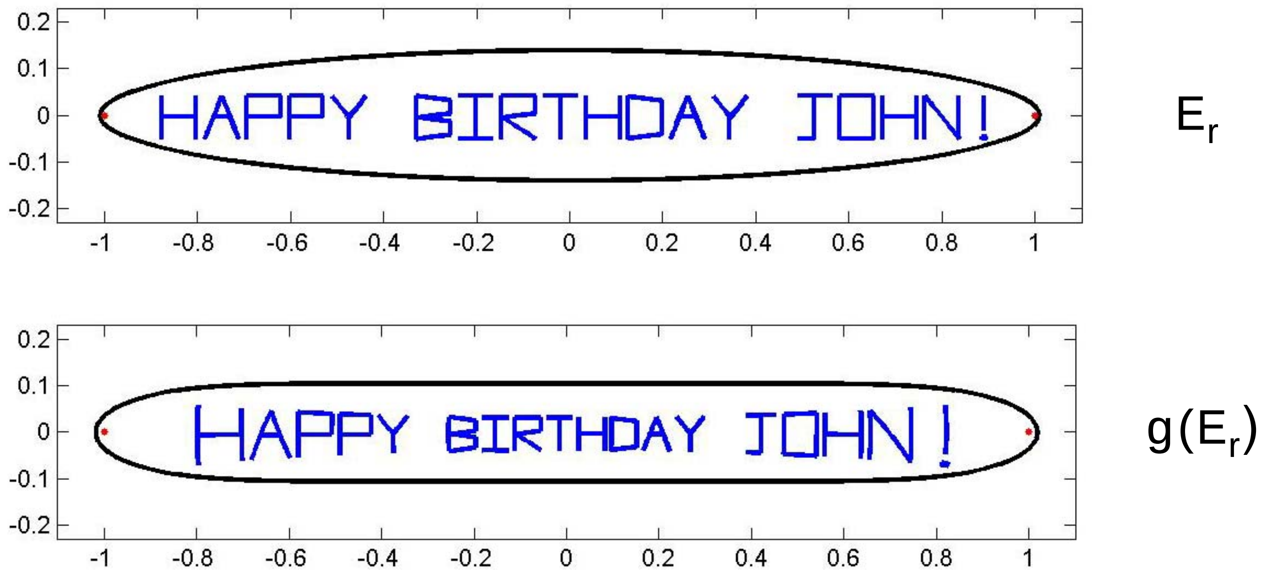
Two kinds of maps g(s) seem particularly useful:

1: Straight-sided regions
(for speeding up quadrature and spectral methods)

2: Regions with pinches
(for adapting to regions of rapid variation)

Idea 1: $g(s)$ maps E_r to a straight-sided region, e.g.

$$g(s) = (40320s+6720s^3+3024s^5+1800s^7+1225s^9)/53089$$



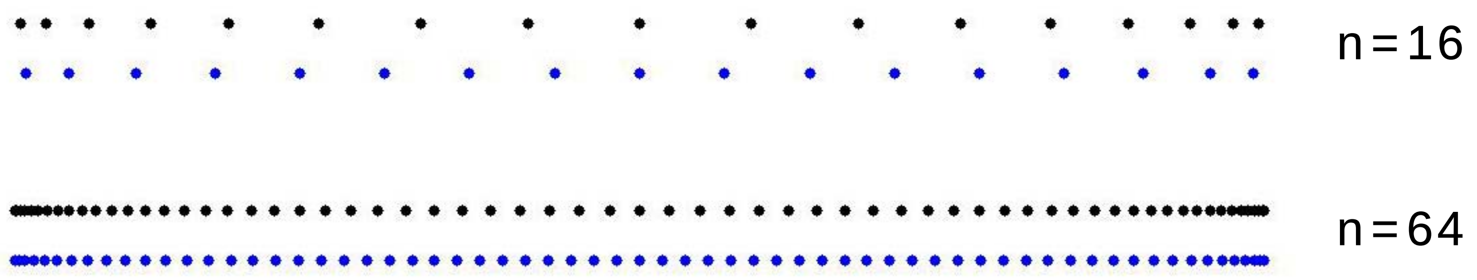
THEOREM. For f analytic in the closed ε-nbhd of [−1,1], ε < 0.8:

Gauss: $|I - I_n| = O((1+\varepsilon)^{-2n})$

Transplanted Gauss: $|I - I_n| = O((1+1.3\varepsilon)^{-2n})$

(Hale & T., SINUM 2008)

GAUSS vs. TRANSPLANTED GAUSS quadrature points

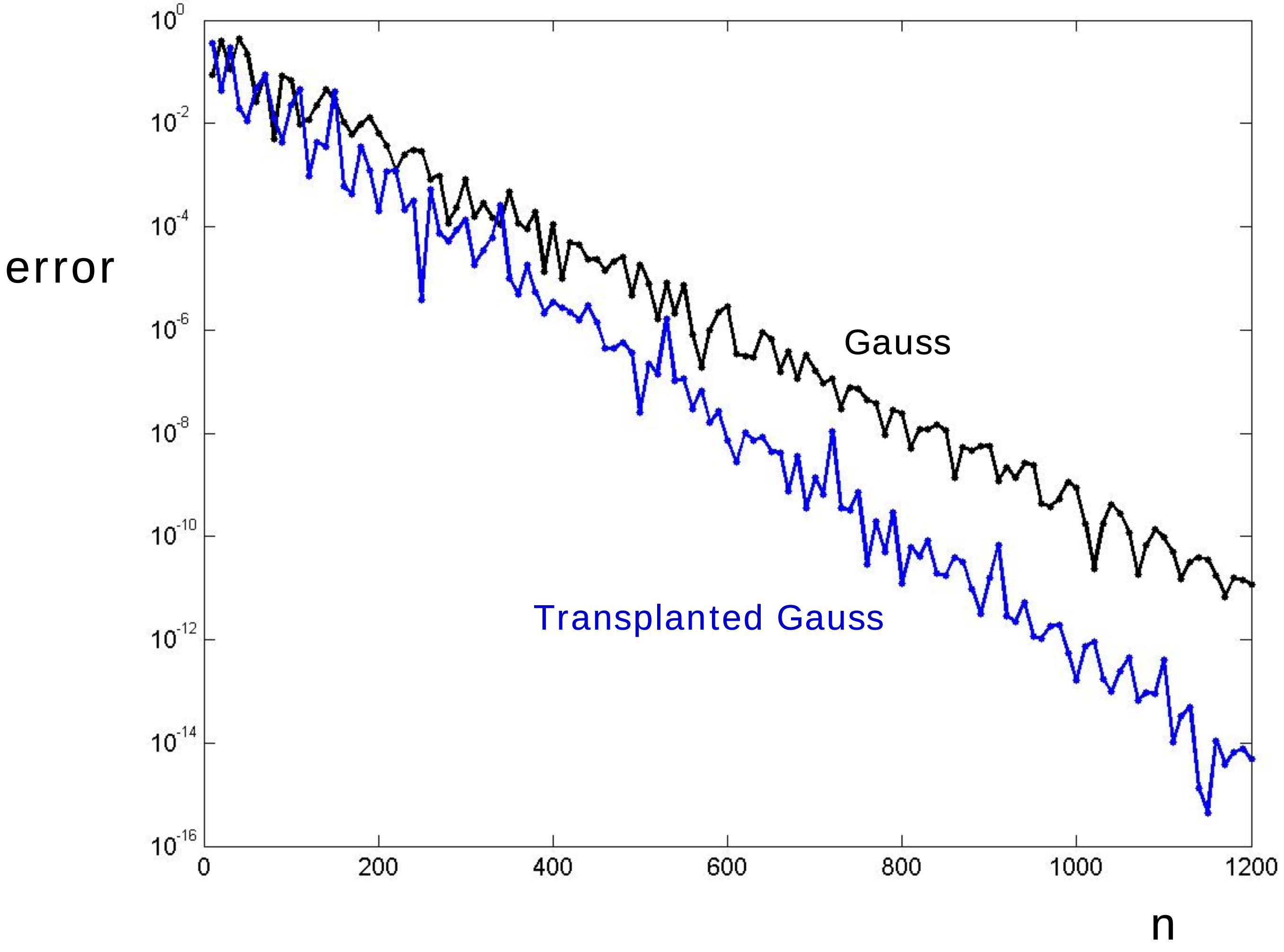


Other choices of g even out the distribution further.
Max possible improvement over Gauss: factor of $\pi/2$.

Such maps applied to spectral methods permit smaller grids and bigger time steps. Especially valuable in 2D and 3D.

Convergence for $f(x) = 1/(\cosh(1)-\cos(100x))$

(analytic in the strip of half-width $a = 0.01$)



RELATED WORK

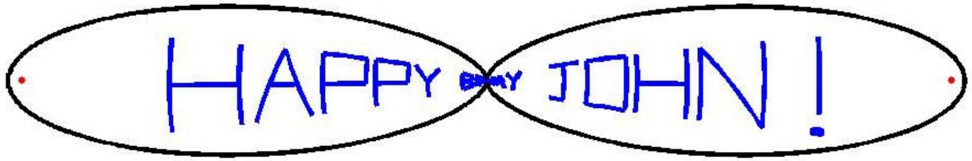
- Gregory formulas: trapezoid rule with endpoint corrections
- Bakhvalov 1967: theoretical results on conformal maps & quadrature
- Kosloff & Tal-Ezer 1993: arcsine change of vars. for spectral methods
- Beylkin, Boyd, Rokhlin & others: prolate spheroidal wave functions
- Alpert 1999: hybrid trapezoid/Gauss quadrature formulas

The last three are roughly as effective as our method in practice.
But they come with no thms about geometric convergence for analytic f.

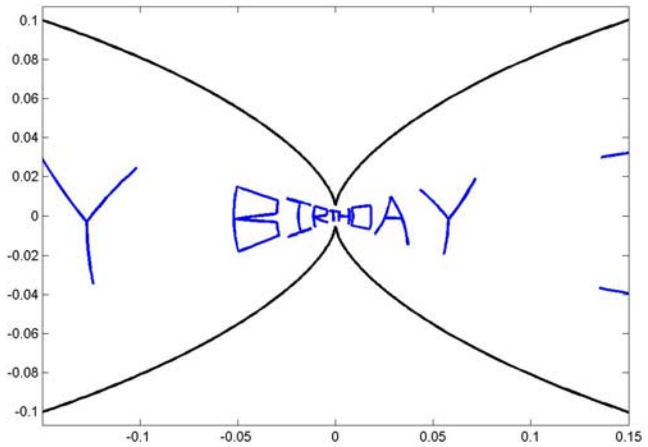
Idea 2: g(s) maps E_r to a region with pinches, e.g.

g(s) = polynomial s.t. g'(s) has zeros near [-1,1]

E_r



$g(E_r)$



closeup

Such maps make possible adaptive spectral method for PDEs — for problems with spikes, fronts, rapid variation.

Wynn Tee and Nick Hale have developed such ideas a long way with the use of Schwarz-Christoffel maps.

(Tee & T. SISC 2006, Hale & Tee SISC to appear.)

RELATED WORK

- Bayliss, Matkowsky & others `87,`89,`90,`92,`95
- Guillard & Peyret `88
- Augenbaum `89
- Kosloff & Tal-Ezer `93
- Mulholland, Huang, Sloan, Qiu `97,`98
- Weideman `99
- Berrut, Baltesnsperger, Mittelmann `00,`01,`02,`04,`05

Conformal maps give better accuracy than these contributions — and with theorems.

MORAL OF THE STORY

It’s not enough for a grid to “look good”.

It must correspond to a transplantation with a wide region of analyticity. And if it does, you get exponential convergence.

Matlab demos