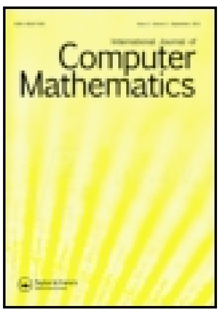

International Journal of Computer Mathematics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gcom20

A family of high order product integration methods for an integral equation of lighthill

Neide Bertoldi Franco [a] & Sean Mc Kee [b]
[a] ICMSC—USP , Sã o Carlos, SP, Brazil
[b] Computing Laboratory , 8-11 Keble Road, Oxford, England
Published online: 19 Mar 2007.

Intern. J. Computer Math., 1985, Vol. 18, pp. 173–184
0020-7160/85 1802-0173 $18.50/0

A Family of High Order Product Integration Methods for an Integral Equation of Lighthill

NEIDE BERTOLDI FRANCO
ICMSC—USP, São Carlos, SP, Brazil

and

SEAN McKEE
Computing Laboratory, 8–11 Keble Road, Oxford, England

(*Received November, 1983; in final form August, 1984*)

Lighthill has derived a nonlinear singular Volterra integral equation to describe the temperature distribution of the surface of a projectile moving through a laminar boundary layer at high Mach numbers. This paper presents high order product integration methods for its numerical solution and analyses their convergence. Numerical results are given.

KEY WORDS: Singular Volterra equation, product integration methods, high accuracy.

C.R. CATEGORY: G.1.9.

1. INTRODUCTION

Consider the Volterra integral equation

$$F(z)^4 = -\frac{1}{2z^{1/2}} \int_0^z \frac{F'(z_1)}{(z^{3/2}-z_1^{3/2})^{1/3}}\, dz_1; \quad F(0)=1, \lim_{z\to\infty} F(z)=0. \qquad (1.1)$$

Lighthill [5] obtained two series solutions, one for small z and one for large z and faired these two curves together.

Following Noble (see Anselone [1], p. 215) using Abel's-type inversion formula it is not difficult to show that (1.1) can be rewritten in the more convenient form

$$F(w)=1-\frac{3\sqrt{3}}{2\pi}\int_0^w \frac{z[F(z)]^4}{(w^{3/2}-z^{3/2})^{2/3}}dz. \tag{1.2}$$

Franco, McKee and Dixon [3] have employed a novel Gronwall lemma to demonstrate the convergence of a simple product integration method for solving (1.2). The object of this paper is to present a family of high order methods and show that they are convergent.

2. THE NUMERICAL METHOD

Consider the Volterra integral equation

$$y(t)=1-\frac{3\sqrt{3}}{2\pi}\int_0^t \frac{s[y(s)]^4}{(t^{3/2}-s^{3/2})^{2/3}}ds, \quad t\in[0,1]. \tag{2.1}$$

The method we shall propose replaces $[y(s)]^4$ by some high order interpolating polynomial.

Let $t=t_i=ih$ where h is the mesh spacing and is defined to be such that $Nh=t_N=1$.

We define for all $h\in(0,h_0)$, $h_0>0$, the function

$$\Delta_h: C[0,1]\to R^{N+1}$$

such that

$$\Delta_h\phi(t)=(\phi(0),\phi(h),\dots,\phi(1-h),\phi(1))^T.$$

Let $\Phi_h z=0$ define all algorithms such that

$$\Phi_h: R^{N+1}\to R^{N+1}.$$

Rewrite Eq. (2.1) in the form

$$y(t_i)=1-\frac{3\sqrt{3}}{2\pi}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \frac{s[y(s)]^4}{(t_i^{3/2}-s^{3/2})^{2/3}}ds \tag{2.2}$$

and replace $[y(s)]^4$ for $s\in[t_j, t_{j+1}]$ by

$$P_n(s)=\begin{cases}\dfrac{1}{h^n}\displaystyle\sum_{l=-j}^{n-j}\frac{[y(t_{j+l})]^4}{(-1)^{n-j-l}(l+j)!(n-j-l)!}\prod_{\substack{k=-j\\k\neq l}}^{n-j}(s-t_{j+k}), & 0\leqq j\leqq n-1\\[2ex] \dfrac{1}{h^n}\displaystyle\sum_{l=1-n}^{1}\frac{[y(t_{j+l})]^4}{(-1)^{1-l}(l+n-1)!(1-l)!}\prod_{\substack{k=1-n\\k\neq l}}^{1}(s-t_{j+k}), & n\leqq j\leqq i-1.\end{cases}\tag{2.3}$$

We can define the algorithm

$$[\Phi_h y]_i=\begin{cases}y_i-\tilde{y}_i, & 0\leqq i\leqq n-1\\[1ex] y_i-1+\dfrac{3\sqrt{3}}{2\pi}h\displaystyle\sum_{j=0}^{i-1}w_{ij}y_j^4+\frac{3\sqrt{3}}{2\pi}w_{ii}y_i^4, & n\leqq i\leqq N\end{cases}\tag{2.4}$$

where $\tilde{y}_i$, $0\leqq i\leqq n-1$, are given initial values, y_i is an approximation to $y(t_i)$ and

$$w_{ij}=\begin{cases}\dfrac{1}{h^{n+1}}\dfrac{1}{(-1)^n 0!n!}\displaystyle\sum_{l=0}^{n-1}I_l(g_i(s)\pi_{-l,-l,l}(s)), & j=0\\[2ex] \dfrac{1}{h^{n+1}}\Bigg\{\dfrac{1}{(-1)^{n-j}j!(n-j)!}\displaystyle\sum_{l=0}^{n-1}I_l(g_i(s)\pi_{-l,-l+j,l}(s)) & \\ \quad+\displaystyle\sum_{p=0}^{q_1}\frac{1}{(-1)^{n-j+p+1}(j-p-1)!(n-j+p+1)!} & \\ \quad\times I_{n+p}(g_i(s)\pi_{1-n,j-n-p,n+p}(s))\Bigg\} & 1\leqq j\leqq n\\[2ex] \dfrac{1}{h^{n+1}}\displaystyle\sum_{l=j-1}^{q_2}\frac{1}{(-1)^{l-j+1}(n+j-l-1)!(l-j+1)!} & \\ \quad\times I_l(g_i(s)\pi_{1-n,-l+j,l}(s)), & n+1\leqq j\leqq i-1\\[2ex] \dfrac{1}{h^n}\dfrac{1}{(-1)^0 n!0!}I_{j-1}(g_i(s)\pi_{1-n,1,j-1}(s)), & j=i.\end{cases}\tag{2.5}$$

Here we have used and subsequently use the notation

$$g_i(s)=\frac{s}{(t_i^{3/2}-s^{3/2})^{2/3}}$$

$$\pi_{\alpha,\gamma,q}(s)=\prod_{\substack{k=\alpha\\k\neq\gamma}}^{n+\alpha}(s-t_{q+k})$$

$$I_m(F(s))=\int_{t_m}^{t_{m+1}}F(s)\,ds$$

$$q_1=\min\{j-1,\ i-n-1\}$$

and

$$q_2=\min\{i-1,\ j+n-1\}.$$

Also here and henceforth we have used the notation

$$\sum_{j\in\Theta}D_j=0;\ \prod_{j\in\Theta}D_j=1$$

if Θ denotes the empty set.

In matrix notation (2.4) becomes

$$\Phi_h y=y-e+\frac{3\sqrt{3}}{2\pi}(hA_N+B_N)y^4 \tag{2.6}$$

where $y^4=(y_0^4,\ y_1^4,\dots,y_N^4)^T$

$$e=(\tilde{y}_0,\ \tilde{y}_1,\dots,\tilde{y}_{n-1},1,1,\dots,1)^T$$

$$(A_N)_{ij}=\begin{cases}w_{ij}, & n\leqq i\leqq N,\quad 0\leqq j\leqq i-1\\ 0, & \text{otherwise}\end{cases}$$

and

$$B_N=\operatorname{diag}(0,0,\dots,0,\ w_{nn},\dots,w_{NN})\text{ with } n \text{ zeros.}$$

3. CONSISTENCY

We begin by defining what we mean by convergent starting values.

DEFINITION 3.1. The starting values $\tilde{y}_i$, $i=0,1,\ldots,n-1$ are said to be convergent of order $n+1$ if, for $0 \leqq i \leqq n-1$, there exists C_1, independent of h and N, such that

$$|y_i - \tilde{y}_i| \leqq C_1 h^{n+1}.$$

THEOREM 3.2 *The discretization Φ_h is consistent of order $n+1$ if there exists C, independent of h* and N, *such that for all $i \geqq n$ we have*

$$\left\| \left[\Delta_h y(t) - e + \frac{3\sqrt{3}}{2\pi}(hA_N + B_N)\Delta_h[y(t)]^4 \right]_i \right\| \leqq Ch^{n+1}. \tag{3.1}$$

Proof We need only consider

$$\left| \int_0^{t_i} g_i(s)[y(s)]^4\,ds - \sum_{j=0}^{i-1} I_j(g_i(s)P_n(s)) \right|$$

$$= \left| \sum_{j=0}^{i-1} I_j(g_i(s)[f(y(s)) - P_n(s)]) \right|$$

$$\leqq \sum_{j=0}^{n-1} \left| I_j(g_i(s)[f(y(s)) - P_n(s)]) \right|$$

$$+ \sum_{j=n}^{i-1} \left| I_j(g_i(s)[f(y(s)) - P_n(s)]) \right|$$

where $P_n(s)$ is defined by (2.3) and $f(y(s)) = [y(s)]^4$.

Using the error formula for interpolation (e.g. Isaacson and Keller [4], p. 190) we have

$$f(y(s)) - P_n(s) = \begin{cases} \pi^{n}_{-j}{}^{j}(s)\dfrac{f^{(n+1)}(\xi_j^1(s))}{(n+1)!}, & 0 \leqq j \leqq n-1 \\[2ex] \pi_1^1{}_{\;n}(s)\dfrac{f^{(n+1)}(\xi_j^2(s))}{(n+1)!}, & n \leqq j \leqq i-1 \end{cases} \tag{3.2}$$

where $\xi_j^1 \in (t_0, t_n)$ and $\xi_j^2 \in (t_{j-n+1}, t_{j+1})$.

Here we have used and subsequently use the notation

$$\pi_\alpha^\beta(s)=\prod_{k=\alpha}^{\beta}(s-t_{j+k}).$$

Let

$$I_j^1=\left|I_j(g_i(s)[f(y(s))-P_n(s)])\right|,\quad 0\leqq j\leqq n-1$$

and

$$I_j^2=\left|I_j(g_i(s)[f(y(s))-P_n(s)])\right|,\quad n\leqq j\leqq i-1.$$

Then using (3.2) and the fact that for

$$s\in[t_j,t_{j+1}],\ s^{1/2}\leqq t_{j+1}^{1/2}$$

and that for $0\leqq j\leqq n-1$,

$$\max_{s\in[t_j,t_{j+1}]}\left\{\left|\pi_{-j}^{n-j}(s)\right|\right\}=(j+1)!h^{j+1}h(n-j)!h^{n-j-1},$$

we can write

$$I_j^1\leqq t_{j+1}^{1/2}(j+1)!(n-j)!h^{n+1}\frac{\left|f^{(n+1)}(\xi_j^1)\right|}{(n+1)!}\int_{t_j}^{t_{j+1}}\frac{s^{1/2}}{(t_i^{3/2}-s^{3/2})^{2/3}}\,ds$$

where $\xi_j^1\in(t_j,t_{j+1})$.

Hence on evaluation of the integral

$$I_j^1\leqq C't_{j+1}^{1/2}h^{n+3/2}\{(i^{3/2}-j^{3/2})^{1/3}-(i^{3/2}-(j+1)^{3/2})^{1/3}\}.$$

Analogously, for $n\leqq j\leqq i-1$, we obtain

$$I_j^2\leqq C''t_{j+1}^{1/2}h^{n+3/2}\{(i^{3/2}-j^{3/2})^{1/3}-(i^{3/2}-(j+1)^{3/2})^{1/3}\}.$$

Therefore

$$\sum_{j=0}^{n-1} I_j^1 + \sum_{j=n}^{i-1} I_j^2 \leqq Ct_i^{1/2}h^{n+3/2} \sum_{j=0}^{i-1} [(i^{3/2}-j^{3/2})^{1/3}-(i^{3/2}-(j+1)^{3/2})^{1/3}]$$

$$= Ct_i^{1/2}h^{n+3/2}i^{1/2}$$

since

$$\sum_{j=0}^{i-1} [(i^{3/2}-j^{3/2})^{1/3}-(i^{3/2}-(j+1)^{3/2})^{1/3}] = i^{1/2} \tag{3.3}$$

and so

$$\sum_{j=0}^{n-1} I_j^1 + \sum_{j=n}^{i-1} I_j^2 \leqq Ct_N^{1/2}h^{n+3/2}N^{1/2} = Ch^{n+1}$$

since $t_N = Nh = 1$.

This completes the demonstration of consistency.

Before stating a convergence theorem a bound is required on the quadrature weights w_{ij}.

LEMMA 3.2 *The quadrature weights w_{ij} are bounded as follows:*

$$|w_{ij}| \leqq \begin{cases} M, & j=0 \\ M\dfrac{j}{(i^{3/2}-j^{3/2})^{2/3}}, & i \leqq j \leqq i-1 \\ Mh^{1/3}, & j=i \end{cases}$$

where M is some constant independent of i, j and h.

Proof The demonstration of this result is not difficult but is tedious due to the many cases that need to be considered (see Franco [2]).

We are now in a position to prove the convergence of the method.

4. CONVERGENCE

Before proving convergence we require a lemma from Franco, McKee and Dixon [3].

LEMMA 4.1 *Let x_i, $i=0,1,\ldots,N$ be a sequence of real numbers satisfying:*

$$|x_0| \leqq \delta,$$

$$|x_i| \leqq \delta(1+Mh) + Mh \sum_{j=1}^{i-1} \frac{j}{(i^{3/2}-j^{3/2})^{2/3}} |x_j|, \quad i=1,2,\ldots,N \tag{4.1}$$

where $\delta>0$, $M>0$ is independent of h, then

$$|x_i| \leqq \delta(1+Mh)\{1+\tfrac{2}{3}MB(\tfrac{4}{3},\tfrac{1}{3})+\tfrac{4}{9}M^2B(\tfrac{1}{3},\tfrac{1}{3})B(\tfrac{4}{3},\tfrac{2}{3})\}$$

$$\times \exp(\tfrac{4}{9}M^3B(\tfrac{2}{3},\tfrac{1}{3})B(\tfrac{1}{3},\tfrac{1}{3})), \quad i=0,1,\ldots,N. \tag{4.2}$$

THEOREM 4.2 *Suppose that the discretization Φ_h is consistent of order $n+1$ and that the starting values are convergent of order $n+1$. Then the discretization Φ_h is convergent of order $n+1$.*

Proof Using (2.6) and adding and subtracting $(3\sqrt{3}/2\pi)(hA_N+B_N)\Delta_h[y(t)]^4$ and using the triangle inequality results in

$$\begin{aligned}
|[y-\Delta_h y(t)]_i| &\leqq \frac{3\sqrt{3}}{2\pi}|[(hA_N+B_N)(y^4-\Delta_h[y(t)]^4)]_i| \\
&\quad + \left|\left[\Delta_h y(t) - e + \frac{3\sqrt{3}}{2\pi}(hA_N+B_N)\Delta_h[y(t)]^4\right]_i\right| \\
&\leqq \frac{3\sqrt{3}}{2\pi} h \sum_{j=0}^{i-1} |(A_N)_{ij}|\,|[y^4-\Delta_h[y(t)]^4]_j| \\
&\quad + \frac{3\sqrt{3}}{2\pi} \max_{n\leq i\leq N} |(B_N)_{ii}|\,|[y^4-\Delta_h[y(t)]^4]_i| + Ch^{n+1}
\end{aligned}$$

using consistency and convergence of the starting values.

Since y is bounded (Lighthill [5]) the function $f(y)=y^4$ is Lipschitz continuous with Lipschitz constant L, say. Thus

$$|[y-\Delta_h y(t)]_i| \leqq \frac{3\sqrt{3}}{2\pi} Lh \sum_{j=0}^{i-1} |(A_N)_{ij}|\,|[y-\Delta_h y(t)]_j|$$

$$+\frac{3\sqrt{3}}{2\pi} L \max_{n \leqq i \leqq N} |(B_N)_{ii}|\,|[y-\Delta_h y(t)]_i| + Ch^{n+1}.$$

Finally from Lemma 3.1 we have

$$|(A_N)_{ij}| = |w_{ij}| \leqq \begin{cases} M, & j=0 \\ M\dfrac{j}{(i^{3/2}-j^{3/2})^{2/3}}, & 1 \leqq j \leqq i-1 \end{cases}$$

and

$$|(B_N)_{ii}| = |w_{ii}| \leqq Mh^{1/3}, \quad i \geqq n.$$

Therefore for h sufficiently small

$$|[y-\Delta_h y(t)]_i| \leqq \frac{hM'}{(1-M'h^{1/3})} \left\{ |y_0-\Delta_h y(t_0)| \right.$$

$$\left. + \sum_{j=1}^{i-1} \frac{j}{(i^{3/2}-j^{3/2})^{2/3}} |[y-\Delta_h y(t)]_j| \right\} + \delta$$

where $M' = (3\sqrt{3}/2\pi)\, L$ and $\delta = Ch^{n+1}/(1-M'h^{1/3})$.

Since $|y_0 - \Delta_h y(t_0)| \leqq C_1 h^{n+1}$ we have

$$|[y-\Delta_h y(t)]_i| \leqq \delta(1+\hat{M}h) + \hat{M}h \sum_{j=1}^{i-1} \frac{j}{(i^{3/2}-j^{3/2})^{2/3}} |[y-\Delta_h y(t)]_j|$$

where $\hat{M} = M'/(1-M'h^{1/3})$.

Application of Lemma 4.1 leads to the required results.

5. NUMERICAL RESULTS

Equation (2.1) was solved by the product integration methods (2.4) of orders one, two and three. Table 1 displays the values obtained with mesh spacings $h=0.25$, 0.025 and 0.0025 and convergence would appear to be being obtained. In Figure 1, the result with $n=2$ (order 3 method) and $h=0.025$ is presented graphically. It is seen to display good agreement with those obtained by Lighthill's asymptotic methods and his somewhat ad hoc approach of fairing together the two curves, one having been obtained from a small z asymptotic expansion and the other from a large z expansion. These results therefore justify Lighthill's approach.

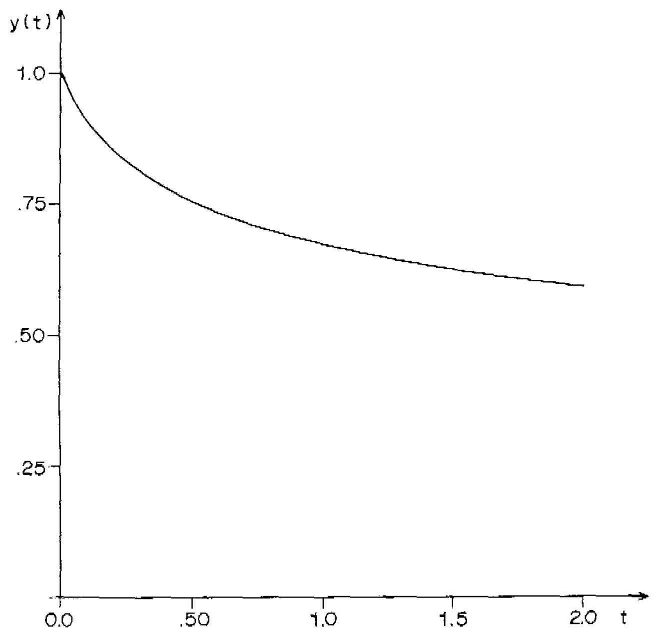

FIGURE 1

Acknowledgements

Both authors would like to acknowledge financial support from Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP), Financiadora de Estudos e Projetos (FINEP) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

TABLE I

h \ t	Order 1 method			Order 2 method			Order 3 method		
	0.75	1.25	1.75	0.75	1.25	1.75	0.75	1.25	1.75
0.25	0.6383	0.5839	0.5433	0.7208	0.6603	0.6197	0.7109	0.6482	0.6081
0.025	0.7044	0.6444	0.6053	0.7061	0.6454	0.6061	0.7061	0.6454	0.6060
0.0025	0.7016	0.6419	0.6030	0.7018	0.6420	0.6030	0.7018	0.6422	0.6031

References

[1] P. M. Anselone, (ed.), *Nonlinear Integral Equations*, University of Wisconsin Press, Madison (1964).

[2] N. B. Franco, *Solução Numérica de Algumas Equações Integrais do Tipo Volterra*, Ph.D. thesis, University of São Paulo, Brazil (1982).

[3] N. B. Franco, S. McKee and J. Dixon, A numerical solution of Lighthill's integral equation of the surface temperature distribution of a projectile, *Matemática Aplicada e Computacional*. Vol. 2, no. 3, pp. 257–271 (1983).

[4] E. Isaacson and H. B. Keller, *The Analysis of Numerical Methods*, John Wiley and Sons Inc, New York, (1966).

[5] M. J. Lighthill, Contributions to the theory of heat transfer through a laminar boundary layer, *Proc. Roy. Soc.* pp. 359–377 (1950).