Part II

NUMERICAL MATHEMATICS

BIT 25 (1985), 624–634

ON THE ORDER OF THE ERROR IN DISCRETIZATION METHODS FOR WEAKLY SINGULAR SECOND KIND VOLTERRA INTEGRAL EQUATIONS WITH NON-SMOOTH SOLUTIONS

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Abstract.

In general, second kind Volterra integral equations with weakly singular kernels of the form $k(t,s)(t-s)^{-\alpha}$ posses solutions which have discontinuous derivatives at $t = 0$. A discrete Gronwall inequality is employed to prove that, away from the origin, the error in product integration and collocation schemes for these equations is of order 2-α.

Keywords: Volterra equations, weakly singular kernels, non-smooth solutions, product integration, error analysis.

1. Introduction.

This note is concerned with the order of the error in numerical schemes for the weakly singular second kind Volterra integral equation

$$(1.1) \qquad y(t) = g(t) + \int_0^t \frac{k(t,s)y(s)}{(t-s)^\alpha} ds, \qquad 0 \leqslant t \leqslant T, \qquad 0 < \alpha < 1.$$

Several authors, including Linz [7], Garey [6], Brunner and Nørsett [3], Cameron and McKee [4], have considered discretization methods for (1.1) and have derived convergence results under the assumption that the solution y is smooth on $[0, T]$.

In most practical examples the solution of (1.1) is not smooth and it can be shown (see, for example, Brunner [2]) that if the given functions g and k satisfy $g \in C^m[0, T]$ and $k \in C^m(D)$, $\quad D := \{(t,s)\colon 0 \leqslant s \leqslant t \leqslant T\}$, then y may be expressed as

$$(1.2) \qquad y(t) = g(t) + \sum_{n=1}^{\infty} \psi_n(t;\alpha)t^{n(1-\alpha)}, \qquad t \in [0, T],$$

where $\psi_n \in C^m[0, T]$ for all n.

Received April 1985. Revised July 1985.

Thus as $t \to 0^+$, $y'(t)$ is discontinuous and it is well-known that in this case the high-order accuracy of product integration and collocation schemes is lost and convergence of order $1-\alpha$ has been proved (see Brunner [2]). However, it has been observed in numerical experiments that as t increases the errors appear to be of order $2-\alpha$ (see te Riele [9], who considers the case of particular practical importance, $\alpha = \frac{1}{2}$).

The purpose of this note is to show by employing a special discrete Gronwall inequality that discretization methods for (1.1) with solution (1.2) satisfy

$$|y(t_i)-y_i| \leqslant C(h^{2-\alpha}t_i^{-\alpha}+h^2), \tag{1.3}$$

where y_i represents an approximation to $y(t_i)$, $t_i = ih$, $0 \leqslant i \leqslant N$, $Nh = T$, and C is a constant independent of h. The bound (1.3) implies the error is of order $2-\alpha$ at the fixed point t_i away from the origin.

The analysis will be illustrated in section 3 by considering the product trapezoidal rule, and in section 5 the results of numerical experiments which confirm the theoretical results for the product trapezoidal rule are given. In section 4 it will be shown how the analysis may also be applied to more general product integration and collocation schemes.

For ease of exposition the linear equation (1.1) will be used; the extension to equations with nonlinear kernels $k(t,s,y)(t-s)^{-\alpha}$ is straightforward provided $k(t,s,y)$ is assumed to be Lipschitz continuous in y. Note that Lubich [8] has established results for the behaviour of the solution near $t = 0$ for nonlinear equations.

2. A discrete Gronwall inequality.

THEOREM 2.1 *Let x_i, $0 \leqslant i \leqslant N$, be a sequence of non-negative real numbers satisfying*

$$x_i \leqslant \chi + \frac{\phi}{(ih)^\alpha} + Mh^{1-\alpha}\sum_{j=0}^{i-1}\frac{x_j}{(i-j)^\alpha}, \qquad 0 \leqslant i \leqslant N, \tag{2.1}$$

where $0 < \alpha < 1$, χ, ϕ are non-negative constants and M is a positive constant, independent of $h\,(h > 0)$, then

$$x_i \leqslant \chi E_{1-\alpha}(M\Gamma(1-\alpha)(ih)^{1-\alpha}) + \frac{\phi\Gamma(1-\alpha)}{(ih)^\alpha}\sum_{n=0}^{\infty}\frac{(M\Gamma(1-\alpha)(ih)^{1-\alpha})^n}{\Gamma((n+1)(1-\alpha))}, \quad 0 \leqslant i \leqslant N, \tag{2.2}$$

where $E_\beta(z)$ is the Mittag-Leffler function defined for $\beta > 0$ by

$$E_\beta(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\beta+1)}.$$

COROLLARY. *If* x_i, $0 \leqslant i \leqslant N$, *satisfies* (2.1) *then for any* $T > 0$ *there exists* $C = C(T)$ *such that*

$$x_i \leqslant C\left(\chi + \frac{\phi}{(ih)^\alpha}\right), \qquad 0 \leqslant i \leqslant N, \tag{2.3}$$

whenever $Nh \leqslant T$.
(Here and elsewhere define $0^{-\alpha} \equiv 1$.)

PROOF. For any sequence g_i, $0 \leqslant i \leqslant N$, define the sequence $\{g_i^{(n)}\}_{n=1}^{\infty}$ as follows:

$$\begin{cases} g_i^{(1)} = g_i, \\ g_i^{(n)} = Mh^{1-\alpha} \displaystyle\sum_{j=0}^{i-1} \frac{g_j^{(n-1)}}{(i-j)^\alpha}, \qquad n \geqslant 2. \end{cases} \tag{2.4}$$

Let $\theta_i = \chi + \phi(ih)^{-\alpha}$, $\quad 0 \leqslant i \leqslant N$.

Then, using the results of Dixon and McKee [5], (2.1) implies

$$x_i \leqslant \sum_{n=1}^{\infty} \theta_i^{(n)}, \qquad 0 \leqslant i \leqslant N,$$

where $\theta_i^{(n)}$, $n \geqslant 1$, is defined by (2.4).

Defining $\chi_i = \chi$, $\quad 0 \leqslant i \leqslant N$, and $\phi_i = \phi(ih)^{-\alpha}$, $\quad 0 \leqslant i \leqslant N$, it follows from (2.4) that $\theta_i^{(n)} = \chi_i^{(n)} + \phi_i^{(n)}$, $\quad n \geqslant 1$.

Hence
$$x_i \leqslant \sum_{n=1}^{\infty} \chi_i^{(n)} + \sum_{n=1}^{\infty} \phi_i^{(n)}, \qquad 0 \leqslant i \leqslant N.$$

By Dixon and McKee [5] $\displaystyle\sum_{n=1}^{\infty} \chi_i^{(n)} \leqslant \chi E_{1-\alpha}(M\Gamma(1-\alpha)(ih)^{1-\alpha})$,

and it only remains to show that

$$\sum_{n=1}^{\infty} \phi_i^{(n)} \leqslant \frac{\phi}{(ih)^\alpha} \Gamma(1-\alpha) \sum_{n=0}^{\infty} \frac{(M\Gamma(1-\alpha)(ih)^{1-\alpha})^n}{\Gamma((n+1)(1-\alpha))}. \tag{2.5}$$

The following inequality will be required:

If $0 < \alpha < 1$ and $\gamma < 1$ then

$$\sum_{j=0}^{i-1} \frac{1}{j^\gamma (i-j)^\alpha} \leqslant i^{1-\alpha-\gamma} \frac{\Gamma(1-\alpha)\Gamma(1-\gamma)}{\Gamma(2-\alpha-\gamma)}. \tag{2.6}$$

A proof of (2.6) may be found in Beesack [1].

Assume inductively that

$$\phi_i^{(n)} \leqslant \phi\Gamma(1-\alpha)\frac{(M\Gamma(1-\alpha))^{n-1}}{\Gamma(n(1-\alpha))}(ih)^{(n-1)-n\alpha}. \tag{2.7}$$

This clearly holds when $n = 1$, and

$$\begin{aligned}\phi_i^{(n+1)} &= Mh^{1-\alpha}\sum_{j=0}^{i-1}\phi_j^{(n)}(i-j)^{-\alpha}\\ &\leqslant \phi\frac{(M\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha))}h^{n-(n+1)\alpha}\sum_{j=0}^{i-1}j^{n(1-\alpha)-1}(i-j)^{-\alpha},\end{aligned}$$

and applying (2.6) with $\gamma = n\alpha-(n-1) < 1$ completes the inductive step. The inequality (2.5) follows from (2.7) and this yields the required bound (2.2).

To obtain (2.3), from (2.2) using $ih \leqslant Nh \leqslant T$,

$$x_i \leqslant \chi E_{1-\alpha}(M\Gamma(1-\alpha)T^{1-\alpha})+\frac{\phi}{(ih)^\alpha}\Gamma(1-\alpha)\sum_{n=0}^{\infty}\frac{(M\Gamma(1-\alpha)T^{1-\alpha})^n}{\Gamma((n+1)(1-\alpha))},$$

and since the series on the right converge uniformly for all T there exists $C = C(T)$ such that

$$x_i \leqslant C\left(\chi+\frac{\phi}{(ih)^\alpha}\right), \qquad 0 \leqslant i \leqslant N.$$

Throughout the rest of this paper, constants C, M, with or without subscripts or superscripts, will denote constants independent of h.

3. The product trapezoidal rule.

The product trapezoidal rule for equation (1.1) is defined as follows:

$$\begin{aligned}y_0 &= g(0)\\ y_i &= g(t_i)+h\sum_{j=0}^{i}w_{ij}k(t_i,t_j)y_j, \qquad 1 \leqslant i \leqslant N,\end{aligned} \tag{3.1}$$

where y_i denotes an approximation to $y(t_i)$, $t_i = ih$, $0 \leqslant i \leqslant N$, $Nh = T$, and

$$w_{i0} = \frac{1}{h^2}\int_0^{t_1} \frac{(t_1 - s)}{(t_i - s)^\alpha}\,ds, \qquad 1 \leqslant i \leqslant N,$$

$$w_{ij} = \frac{1}{h^2}\int_{t_j}^{t_{j+1}} \frac{(t_{j+1} - s)}{(t_i - s)^\alpha}\,ds + \frac{1}{h^2}\cdot\int_{t_{j-1}}^{t_j} \frac{(s - t_{j-1})}{(t_i - s)^\alpha}\,ds, \quad 1 \leqslant j \leqslant i-1, 2 \leqslant i \leqslant N,$$

$$w_{ii} = \frac{1}{h^2}\int_{t_{i-1}}^{t_i} \frac{(s - t_{i-1})}{(t_i - s)^\alpha}\,ds, \qquad 1 \leqslant i \leqslant N.$$

It can be shown that there exists M such that

$$0 < w_{ij} \leqslant M(h(i-j))^{-\alpha}, \qquad 1 \leqslant j \leqslant i \leqslant N. \tag{3.2}$$

The true solution $y(t_i)$ of (1.1) at $t = t_i$ satisfies

$$y(t_i) = g(t_i) + h\sum_{j=0}^{i} w_{ij}k(t_i, t_j)y(t_j) + T_i, \qquad 0 \leqslant i \leqslant N, \tag{3.3}$$

where the quadrature error T_i is given by

$$T_i = \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \left[k(t_i, s)y(s) - \left\{\frac{(t_{j+1} - s)}{h}k(t_i, t_j)y(t_j) + \frac{(s - t_j)}{h}k(t_i, t_{j+1})y(t_{j+1})\right\}\right]\frac{1}{(t_i - s)^\alpha}\,ds, \quad 1 \leqslant i \leqslant N, \text{ and } T_0 = 0. \tag{3.4}$$

From (3.1), (3.2) and (3.3) the error $x_i = |y(t_i) - y_i|$ satisfies

$$x_i \leqslant |T_i| + M\max_{0 \leqslant s \leqslant t \leqslant T} |k(t, s)|h^{1-\alpha}\sum_{j=0}^{i}\frac{x_j}{(i-j)^\alpha}, \quad 0 \leqslant i \leqslant N.$$

Therefore, provided $M\max_{0 \leqslant s \leqslant t \leqslant T} |k(t, s)|h^{1-\alpha} < 1$,

$$x_i \leqslant C_1|T_i| + C_2h^{1-\alpha}\sum_{j=0}^{i-1}\frac{x_j}{(i-j)^\alpha}. \tag{3.5}$$

Consider the quadrature error T_i. Let $\xi > 0$ be given. Let $h_0 > 0$ be fixed and take r to be the smallest integer such that $rh_0 \geqslant \xi$ (r is also fixed).

Define $F(s) = k(t_i, s)y(s)$. Away from the origin $F(s)$ is smooth, therefore, for $s \in [\xi, T]$ we have $|F^{(2)}(s)| \leqslant C < \infty$. Using the error in Lagrange interpolation it follows that for $s \in (t_j, t_{j+1}]$, $j \geqslant r$,

$$|F(s) - \left\{\frac{(t_{j+1}-s)}{h} F(t_j) + \frac{(s-t_j)}{h} F(t_{j+1})\right\}| \leqslant C_3 h^2. \tag{3.6}$$

As $t \to 0^+$, $y(t)$ is non-smooth. Using (1.2) it is straightforward to demonstrate that for $s \in (t_j, t_{j+1}]$, $0 \leqslant j \leqslant r-1$, there exists C_4, such that

$$|F(s) - \left\{\frac{(t_{j+1}-s)}{h} F(t_j) + \frac{(s-t_j)}{h} F(t_{j+1})\right\}| \leqslant C_4 h^{1-\alpha}. \tag{3.7}$$

Using (3.6), (3.7) in (3.4) gives

$$|T_i| \leqslant C_4 h^{1-\alpha} \sum_{j=0}^{r-1} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_i-s)^\alpha} + C_3 h^2 \sum_{j=r}^{i-1} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_i-s)^\alpha}.$$

If $1 \leqslant i \leqslant r$,
$$\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_i-s)^\alpha} = \int_0^{t_i} \frac{ds}{(t_i-s)^\alpha} \leqslant \frac{rh}{(1-\alpha)t_i^\alpha}$$

and for $i > r$,
$$\sum_{j=0}^{r-1} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_i-s)^\alpha} = \int_0^{t_r} \frac{ds}{(t_i-s)^\alpha} < C' \int_0^{t_r} \frac{ds}{t_i^\alpha} = \frac{C' rh}{t_i^\alpha}.$$

It follows, since r is fixed, that

$$|T_i| \leqslant \frac{C_5 h^{2-\alpha}}{t_i^\alpha} + C_6 h^2, \quad 1 \leqslant i \leqslant N. \tag{3.8}$$

Using (3.8) in (3.5) and invoking the corollary to theorem 2.1 it may be deduced that given $\xi > 0$, for all h sufficiently small (with $h \geqslant h_0$ and $Nh = T$), there exists $C = C(T)$ such that

$$|y(t_i) - y_i| \leqslant C(h^{-2\alpha} t_i^{-\alpha} + h^2). \tag{3.9}$$

Hence the order of the error at the fixed point $t_i (h \to 0^+, i \to \infty$ with $t_i = ih$ fixed) away from the origin is $2-\alpha$. In particular, if $\alpha = \frac{1}{2}$, the order of the error as t increases from the origin will be $\frac{3}{2}$.

4. Higher order methods.

In this section it will briefly be shown that similar analysis may be used to prove that methods which are convergent of order $n > 2$ when applied to equation (1.1) under the assumption that the solution of (1.1) is smooth on $[0, T]$ are only convergent of order $2-\alpha$ when the solution of (1.1) is non-smooth as $t \to 0^+$.

Each $i \geqslant n$ may be written uniquely as $i = \nu n + p$ where $\nu \geqslant 1$ and $0 \leqslant p \leqslant n-1$. Then

$$\int_0^{t_i} \frac{k(t_i, s)y(s)}{(t_i - s)^\alpha} ds = \sum_{j=0}^{\nu-2} \int_{t_{jn}}^{t_{(j+1)n}} \frac{k(t_i, s)y(s)}{(t_i - s)^\alpha} ds + \int_{t_{(\nu-1)n}}^{t_i} \frac{k(t_i, s)y(s)}{(t_i - s)^\alpha} ds.$$

For $s \in (t_{jn}, t_{(j+1)n}]$ approximate $k(t_i, s)y(s)$ by an $(n+1)$-point formula and for $s \in (t_{(\nu-1)n}, t_i]$ approximate $k(t_i, s)y(s)$ by an $(n+1+p)$-point formula. This yields

$$\int_0^{t_i} \frac{k(t_i, s)y(s)}{(t_i - s)^\alpha} ds \simeq \sum_{j=0}^{\nu-2} \sum_{k=0}^{n} k(t_i, t_{jn+k})y(t_{jn+k}) \int_{t_{jn}}^{t_{(j+1)n}} l_k(h^{-1}(s - t_{jn}))(t_i - s)^{-\alpha} ds$$

$$\tag{4.1} + \sum_{k=0}^{n+p+1} k(t_i, t_{(\nu-1)n+k})y(t_{(\nu-1)n+k}) \int_{t_{(\nu-1)n}}^{t_i} \bar{l}_k(h^{-1}(s - t_{(\nu-1)n}))(t_i - s)^{-\alpha} ds$$

where $l_k(s)$, $0 \leqslant k \leqslant n$, and $\bar{l}_k(s)$, $0 \leqslant k \leqslant n+p+1$, are fundamental Lagrange polynomials. The integrals are to be evaluated analytically.

Using (4.1) in (1.1) it follows (after rearranging) that a class of discrete methods for (1.1) may be written in the form

$$\begin{aligned} y_0 &= g(t_0) \\ y_i &= \tilde{y}_i, \qquad 1 \leqslant i \leqslant n-1, \\ y_i &= g(t_i) + h \sum_{j=0}^{i} w_{ij} k(t_i, t_j) y_j, \qquad n \leqslant i \leqslant N, \end{aligned}$$

where $\tilde{y}_i$, $1 \leqslant i \leqslant n-1$, are precomputed starting values and the weights w_{ij} involve sums of integrals of the form

$$h^{-1} \int_{t_{jn}}^{t_{(j+1)n}} l_k(h^{-1}(s - t_{jn}))(t_i - s)^{-\alpha} ds \text{ and } h^{-1} \int_{t_{(\nu-1)n}}^{t_i} \bar{l}_k(h^{-1}(s - t_{(\nu-1)n}))(t_i - s)^{-\alpha} ds.$$

It can be shown that the weights satisfy an inequality of the form (3.2) (see Cameron and McKee [4], where examples of methods of this class are given).

The true solution y of (1.1) satisfies

$$\begin{aligned} y(t_i) &= \tilde{y}_i + T_i, \qquad 1 \leqslant i \leqslant n-1, \\ y(t_i) &= g(t_i) + h \sum_{j=0}^{i-1} w_{ij} k(t_i, t_j) y(t_j) + T_i, \qquad n \leqslant i \leqslant N, \end{aligned}$$

where, for $1 \leqslant i \leqslant n-1$, T_i is the error in the ith starting value and, for $n \leqslant i \leqslant N$, T_i is the quadrature error at $t = t_i$.

For h sufficiently small the error $x_i = |y(t_i) - y_i|$ again satisfies the discrete inequality

$$x_i \leqslant C_1|T_i| + C_2 h^{1-\alpha} \sum_{j=0}^{i-1} x_j(i-j)^{-\alpha}, \qquad 0 \leqslant i \leqslant N.$$

Let the starting values be such that

$$|T_i| = |y(t_i) - \tilde{y}_i| \leqslant C_3 h^q, \qquad 1 \leqslant i \leqslant n-1 \qquad (q > 0).$$

(Note that if the product trapezoidal rule is used to compute the starting values then $|T_i| \leqslant C_3 h^{2-\alpha} t_i^{-\alpha}$, $\quad 1 \leqslant i \leqslant n-1$.)

Let r be defined as in section 3. By choosing h_0 sufficiently small it may be assumed with no loss of generality that $n < r$. Set $F(s) = k(t_i, s)y(s)$. Following the analysis of section 3 it follows that for $s \in (t_{jn}, t_{(j+1)n}]$

$$|F(s) - \sum_{k=0}^{n} F(t_{jn+k}) l_k(h^{-1}(s - t_{jn}))| \leqslant \begin{cases} C_4 h^{1-\alpha}, & jn < r \\ C_5 h^{n+1}, & jn \geqslant r. \end{cases}$$

It may then be deduced that for $i \geqslant n$

$$|T_i| \leqslant C_6 h^{2-\alpha} t_i^{-\alpha} + C_7 h^{n+1}.$$

Thus

$$|T_i| \leqslant C_6 h^{2-\alpha} t_i^{-\alpha} + C_8 h^{\mu}, \qquad 1 \leqslant i \leqslant N,$$

where $\mu = \min(q, n+1)$.

Invoking the corollary to theorem 2.1 yields

$$|y(t_i) - y_i| \leqslant C(h^{2-\alpha} t_i^{-\alpha} + h^{\mu})$$

and consequently if $\mu \geqslant 2$ the order of the error at $t = t_i$ away from the origin is again $2 - \alpha$.

Consider also a collocation scheme.

Given collocation parameters $0 < \eta_0 < \eta_1 < \ldots < \eta_m \leqslant 1$ define

$$Q_N := \{t = \hat{t}_{i\sigma} : \hat{t}_{i\sigma} = t_i + \eta_\sigma h, \qquad 0 \leqslant \sigma \leqslant m, \qquad 0 \leqslant i \leqslant N-1\}.$$

Let $S(m; N)$ be the space of piecewise polynomials of degree m defined by

$$S(m; N) := \{u : u|_{\sigma_i} \in \Pi_m, \qquad 0 \leqslant i \leqslant N-1\}$$

where

$$\sigma_0 = [t_0, t_1], \ \sigma_i = (t_i, t_{i+1}], \qquad 1 \leqslant i \leqslant N-1.$$

The solution of (1.1) will be approximated by an element $u \in S(m;N)$: this element will be required to satisfy (1.1) on the finite set Q_N, i.e.

$$u(t) = g(t) + \int_0^t \frac{k(t,s)u(s)}{(t-s)^\alpha}\,ds, \qquad t \in Q_N. \tag{4.2}$$

(See, for example, Brunner [2].)

Since $u|_{\sigma_i} \in \Pi_m$, u may be represented as

$$u(t) = \sum_{\sigma=0}^{m} u(\hat{t}_{i\sigma})\hat{l}_\sigma(h^{-1}(t-t_i)), \qquad t \in \sigma_i, \qquad 0 \leqslant i \leqslant N-1,$$

where $\hat{l}_\sigma(t) = \prod_{\tau \neq \sigma} ((t-\eta_\tau)/(\eta_\sigma - \eta_\tau))$, $\quad 0 \leqslant \sigma \leqslant m$.

Hence the collocation equation (4.2) may be expressed in the form

$$u(\hat{t}_{i\sigma}) = g(\hat{t}_{i\sigma}) + h \sum_{j=0}^{i-1} \sum_{\tau=0}^{m} u(\hat{t}_{j\tau}) h^{-1} \int_{t_j}^{t_{j+1}} k(\hat{t}_{i\sigma}, s)\hat{l}_\tau(h^{-1}(s-t_j))(\hat{t}_{i\sigma}-s)^{-\alpha}\,ds$$

$$+ h \sum_{\tau=0}^{m} u(\hat{t}_{i\tau})h^{-1} \int_{t_i}^{t_{i\sigma}} k(\hat{t}_{i\sigma},)\hat{l}_\tau(h^{-1}(s-t_i))(\hat{t}_{i\sigma}-s)^{-\alpha}ds, \tag{4.3}$$

$0 \leqslant \sigma \leqslant m$, $0 \leqslant i \leqslant N-1$.

The solution y of (1.1) satisfies

$$y(\hat{t}_{i\sigma}) = g(\hat{t}_{i\sigma}) + h \sum_{j=0}^{i-1} \sum_{\tau=0}^{m} y(\hat{t}_{j\tau}) h^{-1} \int_{t_j}^{t_{j+1}} k(\hat{t}_{i\sigma}, s)\hat{l}_\tau(h^{-1}(s-t_j))(\hat{t}_{i\sigma}-s)^{-\alpha}\,ds$$

$$+ h \sum_{\tau=0}^{m} y(\hat{t}_{i\tau})h^{-1} \int_{t_i}^{t_{i\sigma}} k(\hat{t}_{i\sigma}, s)\hat{l}_\tau(h^{-1}(s-t_i))(\hat{t}_{i\sigma}-s)^{-\alpha}ds + T_{i\sigma} \tag{4.4}$$

where

$$T_{i\sigma} = \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{k(\hat{t}_{i\sigma}, s)\{y(s) - \sum_{\tau=0}^{m} y(\hat{t}_{j\tau})\hat{l}_\tau(h^{-1}(s-t_j))\}}{(\hat{t}_{i\sigma}-s)^\alpha}\,ds$$

$$+ \int_{t_i}^{\hat{t}_{i\sigma}} \frac{k(\hat{t}_{i\sigma}, s)\{y(s) - \sum_{\tau=0}^{m} y(\hat{t}_{j\tau})\hat{l}_\tau(h^{-1}(s-t_i))\}}{(\hat{t}_{i\sigma}-s)^\alpha}\,ds, \ 0 \leqslant \sigma \leqslant m,\ 0 \leqslant i \leqslant N-1.$$

Defining r as in section 3, for $s \in \sigma_j$,

$$\left|y(s) - \sum_{\tau=0}^{m} y(\hat{t}_{j\tau})\hat{l}_\tau(h^{-1}(s-t_j))\right| \leqslant \begin{cases} C_1 h^{1-\alpha}, & 0 \leqslant j \leqslant r-1 \\ C_2 h^{m+1}, & r \leqslant j \leqslant N-1. \end{cases}$$

It may then be deduced that for $0 \leqslant \sigma \leqslant m$, $0 \leqslant i \leqslant N-1$,

$$|T_{i\sigma}| \leqslant C_3 h^{2-\alpha}\hat{t}_{i\sigma}^{-\alpha} + C_4 h^{m+1}. \tag{4.5}$$

Let $x_i = \max_{0 \leqslant \sigma \leqslant m} |y(\hat{t}_{i\sigma}) - u(\hat{t}_{i\sigma})|$. It can then be shown using (4.3) and (4.4) that x_i satisfies

$$x_i \leqslant C_5 \max_{0 \leqslant \sigma \leqslant m} |T_{i\sigma}| + C_6 h^{1-\alpha} \sum_{j=0}^{i-1} x_j (i-j)^{-\alpha}, \qquad 0 \leqslant i \leqslant N-1.$$

Full details may be found in Scott [10].

Using (4.5) and invoking the corollary to theorem 2.1 it follows that for some $C = C(T)$

$$\max_{0 \leqq \sigma \leqq m} |y(\hat{t}_{i\sigma}) - u(\hat{t}_{i\sigma})| \leqslant C(h^{2-\alpha} t_i^{-\alpha} + h^{m+1}). \tag{4.6}$$

From the identity

$$y(t) - u(t) = \Big(y(t) - \sum_{\sigma=0}^{m} y(\hat{t}_{i\sigma}) \hat{l}_\sigma(h^{-1}(t-t_i))\Big) + \sum_{\sigma=0}^{m} (y(\hat{t}_{i\sigma}) - u(\hat{t}_{i\sigma})) \hat{l}_\sigma(h^{-1}(t-t_i)), \qquad t \in \sigma_i$$

it may be deduced using (4.6) that, since y is smooth on $(\varepsilon, T]$, for $t \in \sigma_i$, $i \geqslant r$, the error in the collocation approximation satisfies, for some $\hat{C} = \hat{C}(T)$,

$$|y(t) - u(t)| \leqslant \hat{C}(h^{2-\alpha} t_i^{-\alpha} + h^{m+1}), \qquad t \in \sigma_i, \qquad i \geqslant r.$$

Thus the error away from the origin is of order $2-\alpha$.

5. Numerical results.

The convergence of the product trapezoidal rule was tested using equation (1.1) with $k(t,s) = -1$, $g(t) = t^{1/2} + \frac{1}{2}\pi t$ and $\alpha = \frac{1}{2}$; the exact solution in this case is $y(t) = t^{1/2}$.

Table 1 lists the number of correct digits (defined by $-\log_{10}$ (absolute error)) at $t = 0.5, 1, 2$ for $h = 0.1, 0.05, 0.025, 0.01$.

Table 1. *Number of correct digits as defined above.*

t	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.01$
0.5	2.62	3.11	3.59	4.22
1.0	2.98	3.47	3.96	4.53
2.0	3.37	3.38	4.39	*

This table confirms expectations that the order of the error is $\frac{3}{2}$ (note that for a method of order p, halving the stepsize would be expected to yield an increase of $3p/10$ in the number of correct digits).

Additional numerical experiments gave similar results.

6. Concluding remarks.

The discrete Gronwall inequality given in section 2 has been used to analyse the order of the error in the product trapezoidal rule. In a similar manner it has been shown that when the solution of (1.1) is non-smooth the error away from the origin in higher order product integration methods and collocation schemes is of order $2-\alpha$.

Note that if the analysis is applied to the product Euler rule then in place of (3.9) one obtains.

$$|y(t_i)-y_i| \leqslant C(h^{2-\alpha}t_i^{-\alpha}+h)$$

and the order of the error is 1.

Acknowledgements.

The author would like to thank Dr. S. McKee for his interest in this work.

REFERENCES

1. P. R. Beesack, *More generalized discrete Gronwall inequalities.* ZAMM, to appear.
2. H. Brunner, *The approximate solution of Volterra equations with nonsmooth solutions.* SIAM J. Numer. Anal., to appear.
3. H. Brunner and S. P. Nørsett, *Superconvergence of collocation methods for Volterra and Abel equations of the second kind.* Numer. Math. 26 (1981), 347–358.
4. R. F. Cameron and S. McKee, *Product integration methods for second kind Abel integral equations.* J. Comp. Appl. Math. 11 (1984), 1–10.
5. J. A. Dixon and S. McKee, *Weakly singular discrete Gronwall inequalities.* ZAMM, to appear.
6. L. Garey, *The numerical solution of Volterra integral equations with singular kernels.* BIT 14 (1974), 33–39.
7. P. Linz, *Numerical methods for Volterra integral equations with weakly singular kernels.* SIAM. J. Numer. Anal. 6 (1969), 365–374.
8. Ch. Lubich, *Runge-Kutta theory for Volterra and Abel integral equations of the second kind.* Math. Comp. 41 (1983), 87–102.
9. H. J. J. te Riele, *Collocation methods for weakly singular second kind Volterra integral equations with nonsmooth solutions.* IMA J. Numer. Anal. 2 (1982), 437–449.
10. J. A. Scott (née Dixon): *A Unified Analysis of Discretization Methods.* D. Phil. thesis, University of Oxford (1984).