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On the Exact Order of Convergence of Discrete Methods for Volterra-type Equations

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Simple sufficient conditions are given for a wide class of discrete methods for Volterra-type equations to be convergent of order exactly s.

1. Introduction

To compare the solution $y \in X$ of an operator equation with the solution $y^h \in X^h$ of a corresponding discretization, a linear restriction operator $r^h: X \to X^h$ is introduced (Aubin [1], Noble [5]). The sequence $\{y^h\}_{h \in J}$ (with $J \subset \mathbb{R}^+$ and $\inf J = 0$) converges to y if

$$\|r^h y - y^h\|_h \to 0 \quad \text{as} \quad h \to 0,$$

where $\|\bullet\|_h$ denotes a norm on X^h. The convergence is *of order at least s* if, for all $h \in J$ sufficiently small,

$$\|r^h y - y^h\|_h \leq Ch^s,$$

for some constant $C > 0$, independent of h. Convergence will be *of order exactly s* if it is of order at least s and if there exists no $\gamma > 0$ such that, for some $C_\gamma > 0$ independent of h,

$$\|r^h y - y^h\|_h \leq C_\gamma h^{s+\gamma}.$$

The aim of this note is to give sufficient conditions, which are straightforward to verify, for a class of methods for Volterra-type equations to be convergent of order exactly s. Here the term 'Volterra-type equations' encompasses initial-value ordinary differential equations, Volterra integral equations of the first and second kind, and Volterra integrodifferential equations. The notation and concepts employed will generalize those of Dixon & McKee [3].

2. The exact order of convergence

First we introduce the discrete space X^h and the class of discrete methods that will be considered.

DEFINITION 2.1 Let T and h_0 be given, with $T/h_0 = N_0 \in \mathbb{N}$, and define

$$J := \{h : h = T/N,\ N \in \mathbb{N},\ N > N_0\}.$$

For each $h \in J$, let $\Omega^h = \{0, \ldots, N\}$. Define

$$X^h := \{x^h = (x_0, \ldots, x_N) : x_i \in R_i\}$$

where each R_i $(i \in \Omega^h)$ is a real Banach space with norm $|\bullet|_i$. Define the norm $\|\bullet\|_h$ on X^h by

$$\|x^h\|_h = \max\{|x_i|_i : i \in \Omega^h\}.$$

In the following, k_{ij} and $k_{ij}^{(n)}$ $(i = 0, \ldots, N;\ j = 0, \ldots, i)$ will denote respectively the components of a discrete Volterra kernel and the nth discrete iterated Volterra kernel (see Dixon & McKee [3]), and, for $|hk_{ii}| < 1$, the quantity $\hat{k}_{ij}$ will be defined by

$$\hat{k}_{ij} = \begin{cases} k_{ij}/(1 - hk_{ii}) & (j < i), \\ 0 & (j = i). \end{cases}$$

DEFINITION 2.2 A discretization will be said to be *expressible in discrete fundamental form* if it can be expressed as an equation of the form

$$y^h = g^h + H^h(y^h), \tag{2.1}$$

where $g^h \in X^h$ is known and $H^h : X^h \to X^h$ satisfies

$$\big(|H^h(y^h) - H^h(z^h)|\big)_i \leq (K^h\,|y^h - z^h|)_i, \tag{2.2}$$

for all $i \in \Omega^h$ and all $y^h, z^h \in X^h$, where

$$|y^h| = (|y_0|_0, \ldots, |y_N|_N), \qquad (K^h\,|y^h|)_i = h \sum_{j=0}^{i} k_{ij}(|y^h|)_j \quad (i \in \Omega^h)$$

and the discrete Volterra kernel (k_{ij}) satisfies, for all $h \in J$ sufficiently small, the bounds $0 \leq hk_{ii} < 1$ for each $i \in \Omega^h$, and the conditions:

(DI) $\hat{k}_{ij} > 0$ for $0 \leq j < i \leq N$;

(DII) $h\sum_{j=0}^{i-1} \hat{k}_{ij}$ is bounded independently of h for each $i \in \Omega^h$;

(DIII) there exists some $\mu \in \mathbb{N}$ such that the μth discrete iterated kernel $\hat{k}_{ij}^{(\mu)}$ of $\hat{k}_{ij}$ is bounded in i and j independently of h.

This is a generalization of the discrete fundamental form introduced by Dixon & McKee [3]. It is straightforward to apply the Banach fixed-point theorem to show that a discretization expressible in discrete fundamental form possesses a unique solution $y^h \in X^h$.

For Volterra-type equations, there is an extensive class of methods expressible in discrete fundamental form: this is illustrated by Scott [6].

The conditions DI–DIII will be referred to as generalized (zero) stability conditions. For, if g^h in equation (2.1) is perturbed by an amount $\delta g^h \in X^h$, then, by employing Theorem 3.2 of Dixon & McKee [3] (see also Theorem 5.1 of [4]), it can be shown that the change δy^h in y^h satisfies, for all h sufficiently small, the bound

$$\|\delta y^h\|_h \leq C_U\,\|\delta g^h\|_h, \tag{2.3}$$

where $C_U > 0$ is independent of h. That is, the discrete fundamental form is stable (see also Baker [2]).

For convergence, a concept of consistency is required.

DEFINITION 2.3 Let $y \in X$ be the (unique) solution of the Volterra-type equation which is to be solved numerically, and assume a linear restriction operator $r^h : X \to X^h$ has been chosen. Consider a discretization which is expressible in the discrete fundamental form (2.1). The *consistency error* θ^h of the discrete fundamental form is defined to be

$$\theta^h := r^h y - g^h - H^h(r^h y). \tag{2.4}$$

The discrete fundamental form is *consistent* (resp. *consistent of order* s) if

$$\|\theta^h\|_h \to 0 \quad \text{as} \quad h \to 0$$

(resp. $\|\theta^h\|_h \leq Ch^s$, for some $C > 0$ independent of h).

The discretization will further be said to be *optimally consistent of order s* if there exist positive real numbers $C_i(h)$ satisfying (i)

$$C_i(h) < C$$

for some C independent of h, for all $i \in \Omega^h$, and (ii)

$$\max\{C_i(h) : i \in \Omega^h\} \geq \bar{C} > 0,$$

for some $\bar{C}$ independent of h, such that

$$|(\theta^h)_i|_i = C_i(h)h^s. \tag{2.5}$$

Note that optimal consistency of order s implies consistency of order s but not vice versa. Consequently, optimal consistency is the stronger condition and, as the following theorem shows, it may be used to determine the *exact* order of convergence.

THEOREM 2.1 *Let $y \in X$ be the (unique) solution of the underlying Volterra-type equation and let $r^h : X \to X^h$ be a chosen linear restriction operator. Assume that $y^h \in X^h$ is defined by a discretization which is expressible in discrete fundamental form. If the discrete fundamental form satisfies the generalized (zero) stability conditions, then the discretization is convergent if and only if the discrete fundamental form is consistent.*

Moreover, for $h \in J$ sufficiently small, there exist $C_L, C_U > 0$, independent of h, for which

$$C_L \|\theta^h\|_h \leq \|r^h y - y^h\|_h \leq C_U \|\theta^h\|_h,$$

and optimal consistency of order s implies convergence of order exactly s.

Proof. The exact solution $y \in X$ of the underlying Volterra-type equation satisfies the perturbed discrete fundamental form

$$r^h y = g^h + \theta^h + H^h(r^h y). \tag{2.6}$$

Therefore, from (2.3) with $\delta g^h = \theta^h$ and $\delta y^h = r^h y - y^h$, it follows, for h

sufficiently small, that

$$\|r^h y - y^h\|_h \leq C_U \|\theta^h\|_h, \tag{2.7}$$

and consistency of the discrete fundamental form implies convergence.

From (2.6) and (2.1),

$$\theta^h = r^h y - y^h - [H^h(r^h y) - H^h(y^h)].$$

Hence, using (2.2), we obtain

$$\begin{aligned}(|\theta^h|)_i &\leq (|r^h y - y^h|)_i + h \sum_{j=0}^{i} k_{ij}(|r^h y - y^h|)_j \\ &\leq \left(1 + h \max_i \sum_{j=0}^{i} k_{ij}\right) \|r^h y - y^h\|_h .\end{aligned}$$

This holds for all $i \in \Omega^h$, and thus DII gives

$$\|\theta^h\|_h \leq (1 + \hat{C}) \|r^h y - y^h\|_h, \tag{2.8}$$

for some $\hat{C}$, independent of h. Thus convergence implies consistency of the discrete fundamental form. Combining (2.7) and (2.8) yields

$$C_L \|\theta^h\|_h \leq \|r^h y - y^h\|_h \leq C_U \|\theta^h\|_h, \tag{2.9}$$

with $C_L = (1 + \hat{C})^{-1} > 0$.

It remains to prove that optimal consistency of order s implies convergence of order exactly s. Suppose that the discretization is optimally consistent of order s. That is,

$$|(\theta^h)_i|_i = C_i(h) h^s,$$

where $\max\{C_i(h) : i \in \Omega^h\} \geq \bar{C} > 0$, with $\bar{C}$ independent of h, and $C_i(h) \leq C$, with C independent of h, for all $i \in \Omega^h$. Thus

$$\|\theta^h\|_h \leq C h^s,$$

and the upper bound in (2.9) implies convergence of order at least s.

Now suppose that convergence of order $s + \gamma$, with $\gamma > 0$, is possible. Then

$$\|r^h y - y^h\|_h \leq \tilde{C} h^{s+\gamma},$$

for some $\tilde{C} > 0$. The lower bound in (2.9) would then imply

$$\|\theta^h\|_h \leq C_L^{-1} \tilde{C} h^{s+\gamma},$$

which, for sufficiently small h, contradicts optimal consistency.

Note: it is assumed that the discretization does not give the exact solution to the Volterra-type equation. If, for a particular equation, the discretization gave the exact solution, then (2.5) would become

$$\|\theta^h\|_h = 0,$$

and the two-sided bound (2.9) would be replaced by

$$\|r^h y - y^h\|_h = 0.$$

3. Concluding remarks

Theorem 2.1 reduces the task of obtaining the exact order of convergence of a discretization of a Volterra-type equation to three straightforward steps:

(1) express the discretization in discrete fundamental form;

(2) check that the generalized (zero) stability conditions DI–DIII hold;

(3) find the order of optimal consistency.

Performing step (1) and checking that DI and DII hold present no problems. Verifying that DIII holds may be harder for Volterra equations with weakly singular kernels, but, for initial-value ordinary differential equations, DIII holds whenever the usual zero (Dahlquist) stability condition is satisfied.

Finding the order of optimal consistency may require more work (and additional continuity conditions) than merely checking that the method is consistent; the premium for this extra work is the exact order of convergence. This is particularly useful when, for example, cyclic linear multistep methods are employed, either to solve ordinary differential equations or to generate quadrature rules for solving Volterra integral equations. In this case, the order of convergence may succeed the order of the individual members of the cycle; by determining the order of optimal consistency, the correct order of convergence is found.

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