Matrix square root and interpolation spaces

Mario Arioli and Daniel Loghin

m.arioli@rl.ac.uk

STFC-Rutherford Appleton Laboratory, and University of Birmingham

Outline

- Norms and duality in finite dimensional Hilbert spaces
- Discrete Interpolation Norms
- The continuous case and finite-element approximation
- An example: the biharmonic operator
- Summary and open problems

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Identify the norms for which we have

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Finite dimensional Hilbert spaces and $\mathbb{R}^N$

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  $\forall u \in \mathcal{H} \qquad u = \sum_{i=1}^{\mathbf{N}} u_i \psi_i \qquad u_i \in \mathbf{R} \quad i = 1, \dots, \mathbf{N}$

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- Representation of scalar product in $\mathbb{R}^N$.
  Let $u = \sum_{i=1}^{\mathbf{N}} u_i \psi_i$ and $v = \sum_{i=1}^{\mathbf{N}} v_i \psi_i$.
  Then

$$(u,v) = \sum_{i=1}^{\mathbf{N}} \sum_{j=1}^{\mathbf{N}} u_i v_j (\psi_i, \psi_j) = \mathbf{v}^T \mathbf{H} \mathbf{u}$$

  where $\mathbf{H}_{ij} = \mathbf{H}_{ji} = (\psi_i, \psi_j)$ and $\mathbf{u}, \mathbf{v} \in \mathbf{R}^{\mathbf{N}}$.
  Moreover, $\mathbf{u}^T \mathbf{H} \mathbf{u} > 0 \qquad$ iff $\qquad \mathbf{u} \neq 0$ and, thus $\mathbf{H}$ SPD.

Dual space $\mathcal{H}'$

- $f \in \mathcal{H}' : \mathcal{H} \to \mathbf{R}$ (functional);
- $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) \qquad \forall u, v \in \mathcal{H}$
- $\mathcal{H}'$ is the space of the linear functionals on $\mathcal{H}$

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- Dual vector
  Let $u \in \mathcal{H},\ u \neq 0$, then $\exists f_u \in \mathcal{H}'$ such that

$$f_u(u) = \|u\|_{\mathcal{H}}$$

(Hahn-Banach).

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  $$\mathbf{f} = \frac{\mathbf{H}\mathbf{u}}{\|\mathbf{u}\|_{\mathbf{H}}}$$

  and

  $$\|f_u\|^2_{\mathcal{H}'} = \mathbf{u}^T\mathbf{H}\mathbf{u} = \mathbf{f}^T\mathbf{H}^{-1}\mathbf{f}$$

Linear operator

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The interesting case is $\kappa_H(M)$ independent of N

Interpolation spaces

$$\begin{aligned}\mathcal{H} &= (\mathbf{R}^N, (\mathbf{u},\mathbf{v})_{\mathcal{H}} = \mathbf{u}^T H \mathbf{v}) \\ \mathcal{M} &= (\mathbf{R}^N, (\mathbf{u},\mathbf{v})_{\mathcal{M}} = \mathbf{u}^T M \mathbf{v})\end{aligned}$$

$$(\mathbf{u},\mathbf{v})_{\mathcal{H}} = (\mathbf{u},\mathbf{S}\mathbf{v})_{\mathcal{M}} = (\mathbf{S}\mathbf{u},\mathbf{v})_{\mathcal{M}}$$

$$\mathbf{S} = M^{-1} H$$

S self-adjoint in the good scalar product!

$$\left\{\mathbf{S}\mathbf{x} = \mu\mathbf{x} \quad \Leftrightarrow \quad H\mathbf{x} = \mu M\mathbf{x}\right\} \Rightarrow \mu = \delta^2 > 0$$

$\exists W$ s.t. $\quad M = W^T W, \quad H = W^T \Delta^2 W, \quad \Delta \quad$ diagonal

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Interpolation spaces

$$\boldsymbol{\Lambda} = W^{-1}\Delta W \qquad \boldsymbol{\Lambda}^{1/2} = W^{-1}\Delta^{1/2}W$$

$$\begin{aligned} \mathbf{S} = M^{-1}H &= W^{-1}W^{-T}W^{T}\Delta^{2}W \\ &= W^{-1}\Delta W W^{-1}\Delta W \\ &= \boldsymbol{\Lambda}^{2} \end{aligned}$$

$$M\Lambda = W^{T}WW^{-1}\Delta W^{-T}W^{T}W = \Lambda^{T}M \Rightarrow (\mathbf{u}, \Lambda\mathbf{v})_{\mathcal{M}} = (\Lambda\mathbf{u}, \mathbf{v})_{\mathcal{M}}$$

$$(\Lambda^{1/2}\mathbf{u}, \Lambda^{1/2}\mathbf{u})_{\mathcal{M}} = (\mathbf{u}, \Lambda\mathbf{u})_{\mathcal{M}}$$

Interpolation spaces

$$\Big[\mathcal{H},\mathcal{M}\Big]_{\vartheta} = \Big\{\mathbf{u}\in\mathbf{R}^N; \Big((\mathbf{u},\mathbf{u})_{\mathcal{M}} + (\mathbf{u}, S^{1-\vartheta}\mathbf{u})_{\mathcal{M}}\Big)^{1/2}\Big\}$$

$$\Big[\mathcal{H},\mathcal{M}\Big]_{1/2} = \Big\{\mathbf{u}\in\mathbf{R}^N; \Big((\mathbf{u},\mathbf{u})_{\mathcal{M}} + (\mathbf{u}, \Lambda\mathbf{u})_{\mathcal{M}}\Big)^{1/2}\Big\}$$

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$$||v||_{\vartheta}^2 = ||v||_{H_\vartheta}^2 = v^T\left(M + MS^{1-\vartheta}\right)v$$

$$H_\vartheta = M\left(I + S^{1-\vartheta}\right) = W^T\left(I + \Delta^{2(1-\vartheta)}\right)W$$

Interpolation spaces (duality)

$\mathcal{M}'$ and $\mathcal{H}'$ dual spaces of $\mathcal{M}$ and $\mathcal{H}$

$$\Big[\mathcal{H}, \mathcal{M}\Big]'_{\vartheta} = \Big[\mathcal{M}', \mathcal{H}'\Big]_{1-\vartheta}$$

$$S' = MH^{-1} = W^T \Delta^{-2} W^{-T}$$

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$$S' = MH^{-1} = W^T\Delta^{-2}W^{-T}$$

$$H'_{1-\vartheta} = H^{-1}_{1-\vartheta} = W^{-1}\Delta^{-2\vartheta}W^{-T}$$

Interpolation spaces (∞ dimension case)

- X, Y two Hilbert spaces with $X \subset Y$, X dense and continuously embedded in Y. $\langle \cdot, \cdot \rangle_X, \langle \cdot, \cdot \rangle_Y$ and $\| \cdot \|_X, \| \cdot \|_Y$ the respective norms.
- (Riesz representation theory) $\exists \mathcal{J} : X \to Y$ positive and self-adjoint with respect to $\langle \cdot, \cdot \rangle_Y$ such that $\langle u, v \rangle_X = \langle u, \mathcal{J} v \rangle_Y$.
  $\mathcal{E} = \mathcal{J}^{1/2} : X \to Y$,
- $X = D(\mathcal{E})$ with $\|u\|_X \sim \|u\|_{\mathcal{E}} := \left(\|u\|_Y^2 + \|\mathcal{E}u\|_Y^2 \right)^{1/2}$.
- $\|u\|_\theta := \left(\|u\|_Y^2 + \|\mathcal{E}^{1-\theta} u\|_Y^2 \right)^{1/2}$.
- The *interpolation space of index* θ $[X, Y]_\theta := D(\mathcal{E}^{1-\theta}), \;\; 0 \le \theta \le 1$, with the inner-product $\langle u, v \rangle_\theta = \langle u, v \rangle_Y + \left\langle u, \mathcal{E}^{1-\theta} v \right\rangle_Y$ is a Hilbert space (Lions Magenes 1968).
- $[X, Y]_0 = X$ and $[X, Y]_1 = Y$. If $0 < \theta_1 < \theta_2 < 1$ then

$$X \subset [X, Y]_{\theta_1} \subset [X, Y]_{\theta_2} \subset Y.$$

- $\forall \theta \in (0, 1)$ $\pi \in \mathcal{L}(X; \mathcal{X}) \cap \mathcal{L}(Y; \mathcal{Y}) \Longrightarrow \pi \in \mathcal{L}([X, Y]_\theta; [\mathcal{X}, \mathcal{Y}]_\theta)$.

Interpolation spaces (∞ dimension case)

$\Omega \subset \mathbb{R}^n$ open bounded with smooth boundary Γ and let $\boldsymbol{\alpha}$ denote a multi-index of order m where m is a positive integer

$$H^m(\Omega) = \left\{u : D^{\boldsymbol{\alpha}} u \in L^2(\Omega), \quad |\boldsymbol{\alpha}| \le m\right\} \qquad (H^0(\Omega) = L^2(\Omega))$$

$$H^s(\Omega) := [H^m(\Omega), H^0(\Omega)]_{1-s/m}$$

$H_0^s(\Omega)$ completion of $C_0^\infty(\Omega)$ in $H^m(\Omega)$, where $s > 0$. For $0 \le s_2 < s_1$,

$$\begin{cases} [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta = H_0^{(1-\theta)s_1+\theta s_2}(\Omega) & \text{if } (1-\theta)s_1 + \theta s_2 \neq k + 1/2 \\ [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta = H_{00}^{k+1/2}(\Omega) \subset H_0^{k+1/2} & \text{if } (1-\theta)s_1 + \theta s_2 = k + 1/2 \end{cases}$$

$$H^{-s}(\Omega) = (H_0^s(\Omega))' \quad s > 0$$

If $(1-\theta)s_1 + \theta s_2 = 1/2$

$$[H^{-s_1}(\Omega), H^{-s_2}(\Omega)]_\theta = \left(H_{00}^{1/2}(\Omega)\right)'.$$

Finite-element example

$$H_{00}^{1/2}(\Omega) = [H_0^1(\Omega), L^2(\Omega)]_{1/2}.$$

Let $X_h \subset H_0^1(\Omega), Y_h \subset L^2(\Omega)$. Let $\{\phi_i\}_{1 \le i \le n} \in X_h$ be a spanning set for Y_h and let $L_k \in \mathbb{R}^{n \times n}$ denote the Grammian matrices corresponding to the $\langle \cdot, \cdot \rangle_{H_0^k(\Omega)}$-inner product $(H^0(\Omega) = L^2(\Omega))$:

$$(L_k)_{ij} = \langle \phi_i, \phi_j \rangle_{H_0^k(\Omega)}.$$

$H = L_1, \;\; M = L_0$ and

$$H_{1/2,h} = L_0 \Big(I + (L_0^{-1} L_1)^{1/2} \Big)$$

Moreover, we have

$$H_{1/2,h} \sim H_{1/2} = L_0 \Big(L_0^{-1} L_1 \Big)^{1/2}$$

Finite-element example

$$\left(H^{1/2}_{00}(\Omega)\right)' = [H^{-(1)}(\Omega), H^0(\Omega)]_{1/2} = \left([H^1_0(\Omega), H^0_0(\Omega)]_{1/2}\right)'.$$

Let X_h, Y_h be defined as above. Let $Y'_h \subset X'_h = \text{span}\,\{\psi_i\}_{1\le i\le n}$ where ψ_i are basis functions dual to ϕ_i, i.e.,$\langle \psi_i, \phi_j\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} = \delta_{ij}$.

$$\text{If } \; Y'_h \ni z = \sum_{i=1}^n \mathbf{z}_i \psi_i = \sum_{i=1}^n \mathbf{w}_i \phi_i, \quad \phi_l = \sum_{i=1}^n K_{li}\psi_i$$

$$\delta_{ij} = \langle \psi_i, \phi_j\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)} = \sum_{i=1}^n K^{-1}_{il} \langle \phi_l, \phi_j\rangle_{H^1_0(\Omega)} = \sum_{i=1}^n K^{-1}_{il} (L_1)_{lj}$$

so that $\mathbf{z} = L_0\mathbf{w}$ and $\|\mathbf{z}\|_{H'_Y} = \|\mathbf{w}\|_{L_0^{-1}}, \quad \|\mathbf{z}\|_{H'_X} = \|\mathbf{w}\|_{L_0 L_1^{-1} L_0}$ and the matrix representation of H'_Y, H'_X are respectively L_0 and $L_0 L_1^{-1} L_0$.

$$H'_{1/2} = L_0 (L_0^{-1} L_1)^{-1/2} = H_{-1/2}.$$

Evaluation of $H_\theta \mathbf{z}$

- Generalised Lanczos
  $H_X V_k = H_Y V_k T_k + \beta_{k+1} H_y \mathbf{v}_{k+1} \mathbf{e}_k^T, \quad V_k^T H_Y V_k = I_k$
  (T_k tridiagonal).

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- $\mathbf{v}_0 = \mathbf{z}$
  $H_\theta \mathbf{z} \approx H_Y V_k T_k^{1-\theta} \mathbf{e}_1 \|\mathbf{z}\|_{H_Y}$ and
  $H_{\theta,h} \mathbf{z} \approx H_Y V_k (I_k + T_k^{1-\theta}) \mathbf{e}_1 \|\mathbf{z}\|_{H_Y}$.

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  $H_{\theta,h}^{-1} \mathbf{z} \approx V_k (I_k + T_k^{1-\theta})^{-1} \mathbf{e}_1 \|\mathbf{z}\|_{H_Y^{-1}}.$
- Alternative: N. Hale, and N. J. Higham and L. N. Trefethen, SIAM J. Numer. Anal.

An example: biharmonic operator

Consider the biharmonic problem in a polygonal convex open domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma = \cup_{i=1}^{K} \Gamma_i$.

$$\begin{cases} \Delta^2 u &= f & \text{in } \Omega, \\ u = \partial u / \partial n &= 0 & \text{on } \Gamma. \end{cases}$$

$$\begin{cases} -\Delta u &= f & \text{in } \Omega, \\ v + \Delta u &= 0 & \text{in } \Omega, \\ u = \partial u / \partial n &= 0 & \text{on } \Gamma. \end{cases}$$

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An example: biharmonic operator

(Pironneau-Glowinski)

(i) $\begin{cases} -\Delta v_0 = f & \text{in } \Omega, \\ v_0 = 0 & \text{on } \Gamma, \end{cases}$ $\begin{cases} -\Delta u_0 = v_0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma, \end{cases}$

(ii) $\mathcal{S}\lambda = \partial u_0 / \partial \nu \quad \text{on } \Gamma,$

(iii) $\begin{cases} -\Delta v_1 = 0 & \text{in } \Omega, \\ v_1 = \lambda & \text{on } \Gamma, \end{cases}$ $\begin{cases} -\Delta u_1 = v_1 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \Gamma, \end{cases}$

An example: biharmonic operator

$\mathcal{S}$ is a boundary operator which is defined on $H^{-1/2}(\Gamma)$ and which induces a bilinear form

$$s(\cdot,\cdot) : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$$

$s(\cdot,\cdot)$ is symmetric, positive-definite and $H^{-1/2}(\Gamma)-$elliptic, i.e.,

$$c_1\|\lambda\|^2_{H^{-1/2}(\Gamma)} \leq s(\lambda,\lambda) \leq c_2\|\lambda\|^2_{H^{-1/2}(\Gamma)}.$$

An example: biharmonic operator

$$\begin{pmatrix} L & Z \\ Z^T & -M_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{v}_B \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ 0 \end{pmatrix}$$

The Schur complement associated with L in the matrix is

$$S = -M_{BB} - Z^T L^{-1} Z.$$

Let $X_h \subset H^1(\Gamma)$ denote the space spanned by the restriction of the basis functions of V_I^h to the boundary Γ. If $\lambda_h \in X_h$ has a vector of coefficients $\boldsymbol{\lambda}$, then

$$s(\lambda_h, \lambda_h) = \boldsymbol{\lambda}^T S \boldsymbol{\lambda}.$$

An example: biharmonic operator

A discrete $H^{-1/2}$-norm on X_h can be defined as a sum of norms corresponding to each *open* segment of the polygonal boundary Γ:

$$\|\lambda_h\|_{H^{-1/2}(\Gamma)} := \left(\sum_{i=1}^{K} \|\lambda_h\|^2_{H^{-1/2}(\Gamma_i)} \right)^{1/2}.$$

In particular, $H^{-1/2}(\Gamma_i) = (H^{1/2}_{00}(\Gamma_i))'$.

$$\|\lambda_h\|^2_{H^{-1/2}(\Gamma)} = \|\boldsymbol{\lambda}\|^2_{H_{\{-1/2,h\}}} \quad \left(\text{or} \quad = \|\boldsymbol{\lambda}\|^2_{H_{\{-1/2\}}}\right)$$

for $\lambda_h \in X_h$ where

$$H_{\{-1/2,h\}} = \bigoplus_{i=1}^{K} H'^{\{i\}}_{1/2,h}$$

$$\left(H_{\{-1/2\}} = \bigoplus_{i=1}^{K} H^{\{i\}}_{-1/2}, \qquad H^{\{i\}}_{-1/2} = L_{0,i}(L_{0,i}^{-1} L_{1,i})^{-1/2} \right)$$

An example: biharmonic operator

$$P = \begin{pmatrix} L & Z \\ & P_S \end{pmatrix}$$

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n	m	I	$H_{\{-1/2,h\}}$	$H_{\{-1/2\}}$	$\widehat{H}_{\{-1/2\}}$
84,610	640	26	10	12	12
337,154	1,280	30	9	11	11
1,346,050	2,560	36	9	11	11

FGMRES iterations for model problem .

Open problems

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