Problemi di accuratezza relativi alla soluzione di sistemi sottodeterminati

In onore di Alfonso Laratta

Mario Arioli

m.arioli@rl.ac.uk

CCLRC-Rutherford Appleton Laboratory

Sommario

- Problemi sottodeterminati e loro duali
- Modelli di Roundoff
- Analisi dell'errore e accuratezza
- Generalizzazioni
- Sparsità
- Prospettive future

Problemi sottodeterminati e loro duali

$$\mathbf{A}^T\mathbf{u} = \mathbf{b}$$

$$\mathbf{A} \in \mathbf{R}^{n\times m} \; \mathit{rank}(\mathbf{A}) = m \leq n$$

Problemi sottodeterminati e loro duali

$$\mathbf{A}^T\mathbf{u} = \mathrm{b}$$

$$\mathbf{A} \in \mathbb{R}^{n\times m} \; rank(\mathbf{A}) = m \leq n$$

$$\mathbf{u} = \mathbf{A}^{T+}\mathbf{b} + (\mathbf{I} - \mathbf{P})\mathbf{w}$$

Rutherford Appleton Laboratory

Problemi sottodeterminati e loro duali

$$\mathbf{A}^T\mathbf{u} = \mathbf{b}$$

$$\mathbf{A} \in \mathbb{R}^{n\times m} \ \mathit{rank}(\mathbf{A}) = m \leq n$$

$$\mathbf{u} = \mathbf{A}^{T+}\mathbf{b} + (\mathbf{I} - \mathbf{P})\mathbf{w}$$

- $\mathbf{A}^{T+} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}$

Problemi sottodeterminati e loro duali

$$\mathbf{A}^T\mathbf{u} = \mathbf{b}$$

$$\mathbf{A} \in \mathbb{R}^{n\times m} \; \mathit{rank}(\mathbf{A}) = m \leq n$$

$$\mathbf{u} = \mathbf{A}^{T+}\mathbf{b} + (\mathbf{I} - \mathbf{P})\mathbf{w}$$

- $\mathbf{A}^{T+} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}$
- $\mathbf{P} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ e $\mathbf{w} \in \mathbb{R}^n$

Problemi sottodeterminati e loro duali

Primale

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \frac{1}{2}||\mathbf{u}-\mathbf{q}||^2$$

Problemi sottodeterminati e loro duali

Primale

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \frac{1}{2}||\mathbf{u}-\mathbf{q}||^2$$

Sistema Lagrangiano

$$\begin{bmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{b} \end{bmatrix}.$$

Problemi sottodeterminati e loro duali

Primale

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \frac{1}{2}||\mathbf{u}-\mathbf{q}||^2$$

Sistema Lagrangiano

$$\begin{bmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{b} \end{bmatrix}.$$

Duale

$$\min_{\mathbf{x}} \frac{1}{2}(\mathbf{A}\mathbf{x}-\mathbf{q})^T(\mathbf{A}\mathbf{x}-\mathbf{q}) + \mathbf{b}^T\mathbf{x}.$$

Modelli di Roundoff

Let $fl(\cdot)$ denote the result of a floating point computation. We assume that

$$fl(\alpha \Box \beta) = (\alpha \Box \beta)(1 + \delta(\Box, \alpha, \beta)) \; ; \;\; |\delta(\Box, \alpha, \beta)| \leq \varepsilon,$$

where α and β are floating point numbers, ε is the machine precision and $\Box$ is one of $+ - * /$.

Modelli di Roundoff

Let $fl(\cdot)$ denote the result of a floating point computation. We assume that

$$fl(\alpha\Box\beta) = (\alpha\Box\beta)(1+\delta(\Box,\alpha,\beta))\ ;\ \ |\delta(\Box,\alpha,\beta)| \leq \varepsilon,$$

where α and β are floating point numbers, ε is the machine precision and $\Box$ is one of $+ - */$. To a great extent modern computers have arithmetic that satisfies assumption. Furthermore, we assume that the scalar products are accumulated using either extended precision arithmetic or the "Kahan Summation Formula". As a consequence of these assumptions, given $\mathbf{z}$ and $\mathbf{y}$ real vectors of dimension n, we have:

$$fl(\mathbf{z}^T\mathbf{y}) = \mathbf{z}^T\mathbf{y} + \mathbf{z}^T\mathbf{D}\mathbf{y} + s, \quad |\mathbf{D}| \leq 3\varepsilon\mathbf{I},\ \ |s| \leq \mathcal{O}(n\varepsilon^2|\mathbf{z}|^T|\mathbf{y}|).$$

Analisi dell'errore

- Householder e Givens

Analisi dell’errore

- Householder e Givens
  (A., Laratta Numer. Math 85)

Analisi dell'errore

- Householder e Givens
  (A., Laratta Numer. Math 85)

$$(\mathbf{A}+\mathbf{F}) = \widehat{\mathbf{H}} \left[\begin{array}{c} \overline{\mathbf{R}} \\ 0 \end{array} \right],$$

$$\|\mathbf{F}\|_F \leq c_1 m \varepsilon \|\mathbf{A}\|_F.$$

Analisi dell’errore

- Householder e Givens
  (A., Laratta Numer. Math 85)
- Gram-Schmidt e modified Gram-Schmidt

Analisi dell’errore

- Householder e Givens
  (A., Laratta Numer. Math 85)
- Gram-Schmidt e modified Gram-Schmidt
  (A., Laratta LAA 86), (Björck, Paige SIMAX 92), (Giraud, Langou, Rozloznik, van den Eshof 04)

CCLRC Rutherford Appleton Laboratory

Analisi dell'errore

- Householder e Givens
  (A., Laratta Numer. Math 85)
- Gram-Schmidt e modified Gram-Schmidt
  (A., Laratta LAA 86), (Björck, Paige SIMAX 92), (Giraud, Langou, Rozloznik, van den Eshof 04)

$$(\mathbf{A} + \mathbf{F}) = \bar{\mathbf{Q}} \begin{bmatrix} \overline{\mathbf{R}} \\ 0 \end{bmatrix},$$

$$\|\mathbf{F}\|_2 \leq c_1 m \varepsilon \|\mathbf{A}\|_2, \; \|\mathbf{I} - \bar{\mathbf{Q}}^T \bar{\mathbf{Q}}\|_2 \leq c_2 \kappa \varepsilon, \; \kappa = \sigma_1 / \sigma_m \, .$$

Analisi dell'errore

- Householder e Givens
  (A., Laratta Numer. Math 85)
- Gram-Schmidt e modified Gram-Schmidt
  (A., Laratta LAA 86), (Björck, Paige SIMAX 92), (Giraud, Langou, Rozloznik, van den Eshof 04)
- Gauss (Björck, Duff LAA 80)

CCLRC Rutherford Appleton Laboratory

Analisi dell'errore

- Householder e Givens
  (A., Laratta Numer. Math 85)
- Gram-Schmidt e modified Gram-Schmidt
  (A., Laratta LAA 86), (Björck, Paige SIMAX 92), (Giraud, Langou, Rozloznik, van den Eshof 04)
- Gauss (Björck, Duff LAA 80)

$$(\mathbf{A} + \delta \mathbf{A}) = \bar{\mathbf{L}} \begin{bmatrix} \bar{\mathbf{U}} \\ 0 \end{bmatrix},$$

$$|\delta \mathbf{A}| \leq c_1 m \varepsilon (|\mathbf{A}| + |\bar{\mathbf{L}}|\mathbf{E}_1|\bar{\mathbf{U}}|) + \mathcal{O}(\varepsilon^2) \qquad \mathbf{E}_1 = \begin{bmatrix} \mathbf{I}_m \\ 0_{n-m,m} \end{bmatrix}.$$

Analisi dell'errore

- Householder e Givens
  (A., Laratta Numer. Math 85)
- Gram-Schmidt e modified Gram-Schmidt
  (A., Laratta LAA 86), (Björck, Paige SIMAX 92), (Giraud, Langou, Rozloznik, van den Eshof 04)
- Gauss (Björck, Duff LAA 80)

Analisi dell'errore (A., Laratta 85)

$$\mathbf{HA} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

Analisi dell'errore (A., Laratta 85)

$$\mathbf{HA} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

$$\mathbf{R}^T \mathbf{d} = \mathbf{b}$$

Analisi dell'errore (A., Laratta 85)

$$\mathbf{HA} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

$$\mathbf{R}^T\mathbf{d} = \mathbf{b}$$

$$\mathbf{Hq} = \begin{bmatrix} \mathbf{Q}^T \\ \mathbf{S}^T \end{bmatrix} \mathbf{q} = \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{bmatrix}$$

Analisi dell'errore (A., Laratta 85)

$$\mathbf{H}\mathbf{A} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

$$\mathbf{R}^T\mathbf{d} = \mathbf{b}$$

$$\mathbf{H}\mathbf{q} = \begin{bmatrix} \mathbf{Q}^T \\ \mathbf{S}^T \end{bmatrix} \mathbf{q} = \begin{bmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{bmatrix}$$

$$\mathbf{u}^* = \mathbf{H}^T \begin{bmatrix} \mathbf{d} \\ \mathbf{h}_2 \end{bmatrix}$$

Analisi dell'errore

Sia $\bar{\mathbf{u}}$ la soluzione calcolata con il precedente algoritmo

$$\exists \mathbf{E} \in \mathbf{R}^{n \times m}, \mathbf{e} \in \mathbf{R}^n, \tilde{\mathbf{u}} \in \mathbf{R}^n$$

$\tilde{\mathbf{u}}$ soluzione di

$$\min_{(\mathbf{A}+\mathbf{E})^T \mathbf{u} = \mathbf{b}} \frac{1}{2} ||\mathbf{u} - (\mathbf{q} + \mathbf{e})||$$

$$\|\mathbf{E}\|_F \leq c(n,m)\|\mathbf{A}\|_F \varepsilon + \mathcal{O}(\varepsilon^2) \qquad \|\mathbf{e}\| \leq c'(n,m)\|\mathbf{q}\|\varepsilon + \mathcal{O}(\varepsilon^2)$$

$$\frac{\|\bar{\mathbf{u}} - \tilde{\mathbf{u}}\|}{\|\bar{\mathbf{u}}\|} \leq c''(n,m)\varepsilon + \mathcal{O}(\varepsilon^2)$$

Analisi dell'errore: sistema Lagrangiano

Null Space Algorithm:

$$\begin{bmatrix} \mathbf{h} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{H}^T\mathbf{q} \\ \mathbf{R}^{-T}\mathbf{b} \end{bmatrix},$$

$$\mathbf{h}_1 = \mathbf{E}_1^T\mathbf{h} = [\mathbf{I}_m \; 0_{m,n-m}]\,\mathbf{h},$$

$$\mathbf{h}_2 = \mathbf{E}_2^T\mathbf{h} = [0_{n-m,m} \; \mathbf{I}_{n-m}]\,\mathbf{h}.$$

Solve the block lower triangular system:

$$\begin{bmatrix} \mathbf{I}_m & 0 & 0 \\ 0 & \mathbf{I}_{n-m} & 0 \\ \mathbf{I}_m & 0 & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{h}_2 \\ \mathbf{h}_1 \end{bmatrix},$$

and let

$$\mathbf{u} = \mathbf{H} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}.$$

Generalizzazioni

Primale

$M \in \mathbf{R}^{n\times n}$ SPD

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \frac{1}{2}\mathbf{u}^T\mathbf{M}\mathbf{u} - \mathbf{q}^T\mathbf{u}$$

Generalizzazioni

Primale
$M \in \mathbf{R}^{n\times n}$ SPD

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \frac{1}{2}\mathbf{u}^T\mathbf{M}\mathbf{u} - \mathbf{q}^T\mathbf{u}$$

Sistema Lagrangiano

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{b} \end{bmatrix}.$$

Generalizzazioni

Primale
$M \in \mathbf{R}^{n\times n}$ SPD

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \frac{1}{2}\mathbf{u}^T\mathbf{M}\mathbf{u} - \mathbf{q}^T\mathbf{u}$$

Sistema Lagrangiano

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{b} \end{bmatrix}.$$

Duale

$$\min_{\mathbf{x}} \frac{1}{2}(\mathbf{A}\mathbf{x}-\mathbf{q})^T\mathbf{M}^{-1}(\mathbf{A}\mathbf{x}-\mathbf{q}) + \mathbf{b}^T\mathbf{x}.$$

Generalizzazioni

Primale

$M \in \mathbf{R}^{n\times n}$ SPD

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \frac{1}{2}\mathbf{u}^T\mathbf{M}\mathbf{u} - \mathbf{q}^T\mathbf{u}$$

Sistema Lagrangiano

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{b} \end{bmatrix}.$$

Duale

$$\min_{\mathbf{x}} \frac{1}{2}(\mathbf{A}\mathbf{x}-\mathbf{q})^T\mathbf{M}^{-1}(\mathbf{A}\mathbf{x}-\mathbf{q}) + \mathbf{b}^T\mathbf{x}.$$

$$\mathit{Ker}\,(\mathbf{M}) \cap \mathit{Ker}\,(\mathbf{A}^T) = 0 \quad \Rightarrow \exists!\mathbf{u}$$

Generalizzazioni

Primale

$\mathbf{C} \in \mathbf{R}^{k \times n}$ $m \leq n \leq k$

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \frac{1}{2}||\mathbf{C}\mathbf{u} - \mathbf{d}||^2$$

Generalizzazioni

Primale

$\mathbf{C} \in \mathbf{R}^{k \times n}$ $m \leq n \leq k$

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \frac{1}{2}||\mathbf{C}\mathbf{u} - \mathbf{d}||^2$$

Sistema Lagrangiano

$$\begin{bmatrix} \mathbf{I} & \mathbf{C} & 0 \\ \mathbf{C}^T & 0 & \mathbf{A} \\ 0 & \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{u} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ 0 \\ \mathbf{b} \end{bmatrix}.$$

Generalizzazioni

Primale

$\mathbf{C} \in \mathbf{R}^{k\times n}$ $m \le n \le k$

$$\min_{\mathbf{A}^T\mathbf{u}=\mathbf{b}} \frac{1}{2}||\mathbf{C}\mathbf{u}-\mathbf{d}||^2$$

Sistema Lagrangiano

$$\begin{bmatrix} \mathbf{I} & \mathbf{C} & 0 \\ \mathbf{C}^T & 0 & \mathbf{A} \\ 0 & \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{u} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ 0 \\ \mathbf{b} \end{bmatrix}.$$

$$\mathit{Ker}\left(\mathbf{C}\right) \cap \mathit{Ker}\left(\mathbf{A}^T\right) = 0 \quad \Rightarrow \exists! \mathbf{u}$$

Analisi dell'errore

- Constrained LS

Analisi dell’errore

- Constrained LS
  (Laratta, Zironi JOTA 90), (Galligani, Laratta Computing 94),
  (Galligani, Zanni Computing 00)

Analisi dell'errore

- Constrained LS
  (Laratta, Zironi JOTA 90), (Galligani, Laratta Computing 94), (Galligani, Zanni Computing 00)
- QP

Analisi dell’errore

- Constrained LS
  (Laratta, Zironi JOTA 90), (Galligani, Laratta Computing 94), (Galligani, Zanni Computing 00)
- QP
  (A. SIMAX 00), (A. , Baldini SIMAX 01), (Galligani Zanni Computing 97), (Galligani Zanni BUMI 97)

Constrained LS (Galligani Laratta 94)

$$\mathbf{HA} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

Constrained LS (Galligani Laratta 94)

$$\mathbf{HA} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

$$\mathbf{CH} = [\mathbf{C}_1 \mathbf{C}_2]$$

Constrained LS (Galligani Laratta 94)

$$\mathbf{HA} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

$$\mathbf{CH} = [\mathbf{C}_1 \mathbf{C}_2]$$

$$\mathbf{R}^T \mathbf{y}_1 = \mathbf{b}$$

Constrained LS (Galligani Laratta 94)

$$\mathbf{HA} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

$$\mathbf{CH} = [\mathbf{C}_1 \mathbf{C}_2]$$

$$\mathbf{R}^T \mathbf{y}_1 = \mathbf{b}$$

$$\mathbf{r} = \mathbf{C}_1 \mathbf{y}_1$$

Constrained LS (Galligani Laratta 94)

$$\mathbf{HA} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

$$\mathbf{CH} = [\mathbf{C}_1 \mathbf{C}_2]$$

$$\mathbf{R}^T \mathbf{y}_1 = \mathbf{b}$$

$$\mathbf{r} = \mathbf{C}_1 \mathbf{y}_1$$

$$\bar{\mathbf{d}} = \mathbf{d} - \mathbf{r}$$

Constrained LS (Galligani Laratta 94)

$$\mathbf{HA} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

$$\mathbf{CH} = [\mathbf{C}_1 \mathbf{C}_2]$$

$$\mathbf{R}^T \mathbf{y}_1 = \mathbf{b}$$

$$\mathbf{r} = \mathbf{C}_1 \mathbf{y}_1$$

$$\bar{\mathbf{d}} = \mathbf{d} - \mathbf{r}$$

$$\min_{\mathbf{z}} ||\mathbf{C}_2 \mathbf{z} - \bar{\mathbf{d}}||$$

Constrained LS (Galligani Laratta 94)

$$\mathbf{HA} = \begin{bmatrix} \mathbf{R} \\ 0 \end{bmatrix}$$

$$\mathbf{CH} = [\mathbf{C}_1 \mathbf{C}_2]$$

$$\mathbf{R}^T \mathbf{y}_1 = \mathbf{b}$$

$$\mathbf{r} = \mathbf{C}_1 \mathbf{y}_1$$

$$\bar{\mathbf{d}} = \mathbf{d} - \mathbf{r}$$

$$\min_{\mathbf{z}} ||\mathbf{C}_2 \mathbf{z} - \bar{\mathbf{d}}||$$

$$\mathbf{u}^* = \mathbf{H} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{z} \end{bmatrix}$$

Constrained LS (Galligani Laratta 94)

Sia $\mathbf{u}'$ la soluzione calcolata dall'algoritmo in aritmetica a precisione finita. Se $c(n,m)\sqrt{m}K(\mathbf{A})\varepsilon < 1$, esistono

$$\mathbf{G} \in \mathbf{R}^{k\times n}, \mathbf{F} \in \mathbf{R}^{n\times m}, \mathbf{e} \in \mathbf{R}^{k}$$

tali che $\tilde{\mathbf{u}} \in \mathbf{R}^n$ é soluzione del problema perturbato

$$\min_{(\mathbf{A}+\mathbf{F})^T\mathbf{u}=\mathbf{b}} \frac{1}{2}||(\mathbf{C}+\mathbf{G})\mathbf{u} - (\mathbf{d}+\mathbf{e})||^2$$

con

$$\frac{||\mathbf{u}' - \tilde{\mathbf{u}}||}{||\mathbf{u}'||} \le c'(n,m)\varepsilon + \mathcal{O}(\varepsilon^2)$$

$$\begin{aligned}
||\mathbf{G}|| &\le c_1(n,m)||\mathbf{C}||\varepsilon + \mathcal{O}(\varepsilon^2) \\
||\mathbf{F}|| &\le c_2(n,m)||\mathbf{A}||\varepsilon + \mathcal{O}(\varepsilon^2) \\
||\mathbf{e}|| &\le c_3(k,n-m)(||\mathbf{d}|| + K(\mathbf{A})||\mathbf{u}_m||||\mathbf{C}||)\varepsilon + \mathcal{O}(\varepsilon^2) \\
\mathbf{u}_m &= \mathbf{A}^+\mathbf{b}.
\end{aligned}$$

Constrained LS (Galligani Laratta 94)

Sotto le medesime ipotesi $\mathbf{u}'$ é soluzione del problema perturbato

$$\min_{(\mathbf{A}+\mathbf{F})^T\mathbf{u}=\mathbf{b}+\delta\mathbf{b}} \frac{1}{2}||(\mathbf{C}+\mathbf{G})\mathbf{u}-(\mathbf{d}+\delta\mathbf{d})||^2$$

con

$$\begin{aligned}
||\mathbf{G}|| &\leq c_1(n,m)||\mathbf{C}||\varepsilon + \mathcal{O}(\varepsilon^2) \\
||\mathbf{F}|| &\leq c_2(n,m)||\mathbf{A}||\varepsilon + \mathcal{O}(\varepsilon^2) \\
||\delta\mathbf{b}|| &\leq c_4(n,m)||\mathbf{u}'||||\mathbf{A}||\varepsilon + \mathcal{O}(\varepsilon^2) \\
||\delta\mathbf{d}|| &\leq c_5(n,m)||\mathbf{u}'||||\mathbf{C}||\varepsilon + c_6(k,n-m)(||\mathbf{d}|| + K(\mathbf{A})||\mathbf{u}_m||||\mathbf{C}||)\varepsilon + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

CCLRC Rutherford Appleton Laboratory

Constrained LS (Laratta, Zironi 90)

$$\varepsilon_C = \frac{||\mathbf{G}||}{||\mathbf{C}||}, \quad \varepsilon_A = \frac{||\mathbf{F}||}{||\mathbf{A}||}, \quad \varepsilon_b = \frac{||\delta\mathbf{b}||}{||\mathbf{b}||}, \quad \varepsilon_d = \frac{||\delta\mathbf{d}||}{||\mathbf{d}||}.$$

$$\begin{aligned} \frac{||\mathbf{u}' - \hat{\mathbf{u}}||}{||\hat{\mathbf{u}}||} &\leq K_A(\mathbf{C})^2(\nu(\mathbf{C},\mathbf{A})\varepsilon_A + \varepsilon_C)\rho \\ &+ K_A(\mathbf{C})(\varepsilon_A + \gamma\varepsilon_d) + K^C(\mathbf{A})(\varepsilon_A + \varepsilon_b) \end{aligned}$$

$$\begin{aligned} K_A(\mathbf{C}) &= ||\mathbf{C}||\;||(\mathbf{C}(\mathbf{I}-\mathbf{P}))^+|| \\ K^C(\mathbf{A}) &= ||\mathbf{A}||\;||L_{IC}^+||, \qquad L_{IC}^+ = (\mathbf{I} - (\mathbf{C}(\mathbf{I}-\mathbf{P}))^+\mathbf{C})\mathbf{A}^{T+} \\ \nu(\mathbf{C},\mathbf{A}) &= ||\mathbf{C}L_{IC}^+||\frac{||\mathbf{A}||}{||\mathbf{C}||} \quad \rho = \frac{||\mathbf{d}-\mathbf{C}\hat{\mathbf{u}}||}{||\mathbf{C}||\;||\hat{\mathbf{u}}||}, \quad \gamma = \frac{||\mathbf{d}||}{||\mathbf{C}||\;||\hat{\mathbf{u}}||}. \end{aligned}$$

Constrained LS (Laratta, Zironi 90)

$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$
valori singolari di $\mathbf{C}$ e
$\tau_1 \geq \tau_2 \geq \cdots \geq \tau_{n-m} > \tau_{n-m+1} = \cdots = \tau_n = 0$
valori singolari di $\mathbf{C}(\mathbf{I} - \mathbf{P})$

$$\mu_j \geq \tau_j \geq \mu_{j+m}$$

$$\begin{aligned} \frac{\mu_1}{\mu_{n-m}} &\leq K_A(\mathbf{C}) = \frac{\mu_1}{\tau_{n-m}} \leq \frac{\mu_1}{\mu_n} = K(\mathbf{C}) \\ K(\mathbf{A}) &\leq K^C(\mathbf{A}) \leq K(\mathbf{C})K(\mathbf{A}) \end{aligned}$$

QP

Null Space Algorithm:

$$\begin{aligned}
\begin{bmatrix} \mathbf{h} \\ \mathbf{v} \end{bmatrix} &= \begin{bmatrix} \mathbf{H}^T\mathbf{q} \\ \mathbf{R}^{-T}\mathbf{b} \end{bmatrix}, \\
\mathbf{h}_1 &= \mathbf{E}_1^T\mathbf{h}, \\
\mathbf{h}_2 &= \mathbf{E}_2^T\mathbf{h}.
\end{aligned}$$

Solve the block lower triangular system: $\tilde{\mathbf{M}} = \mathbf{H}^T\mathbf{M}\mathbf{H}$

$$\begin{bmatrix} \mathbf{I}_m & 0 & 0 \\ \tilde{\mathbf{M}}_{12}^T & \tilde{\mathbf{M}}_{22} & 0 \\ \tilde{\mathbf{M}}_{11} & \tilde{\mathbf{M}}_{12} & I_m \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \mathbf{z}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \mathbf{h}_2 \\ \mathbf{h}_1 \end{bmatrix},$$

and let

$$\mathbf{u} = \mathbf{H} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}, \qquad \mathbf{x} = \mathbf{R}^{-1}\mathbf{z}_3.$$

QP

Siano $\bar{\mathbf{u}}$ e $\bar{\mathbf{x}}$ i valori calcolati delle soluzioni $\mathbf{u}$ and $\mathbf{x}$, con il **Null Space Algorithm**. Se $n\varepsilon \ll 1$ esistono le matrici $\delta\mathbf{M}_1 \in \mathbf{R}^{n\times n}$ $\delta\mathbf{A}_1,\ \delta\mathbf{A}_2 \in \mathbf{R}^{n\times m}$, e il vettore $\delta\mathbf{q} \in \mathbf{R}^n$ tali che

$$\begin{bmatrix} \mathbf{M}+\delta\mathbf{M}_1 & \mathbf{A}+\delta\mathbf{A}_1 \\ (\mathbf{A}+\delta\mathbf{A}_2)^T & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} (\mathbf{q}+\delta\mathbf{q}) \\ \mathbf{b} \end{bmatrix}.$$

$$\begin{aligned} ||\delta\mathbf{A}_1||_F &\leq c_2 m\varepsilon||\mathbf{A}||_F + \mathcal{O}(\varepsilon^2), \\ ||\delta\mathbf{A}_2||_F &\leq c_2 m\varepsilon||\mathbf{A}||_F + \mathcal{O}(\varepsilon^2), \\ ||\delta\mathbf{q}||_2 &\leq c_1\varepsilon||\mathbf{q}||_2 + \mathcal{O}(\varepsilon^2), \end{aligned}$$

e

$$||\delta\mathbf{M}_1||_F \leq c(n,m)\varepsilon\|\mathbf{M}\|_2 + \mathcal{O}(\varepsilon^2).$$

Sparsità

M e **A** sparse

Sparsità

M e **A** sparse

- Stokes e Darcy

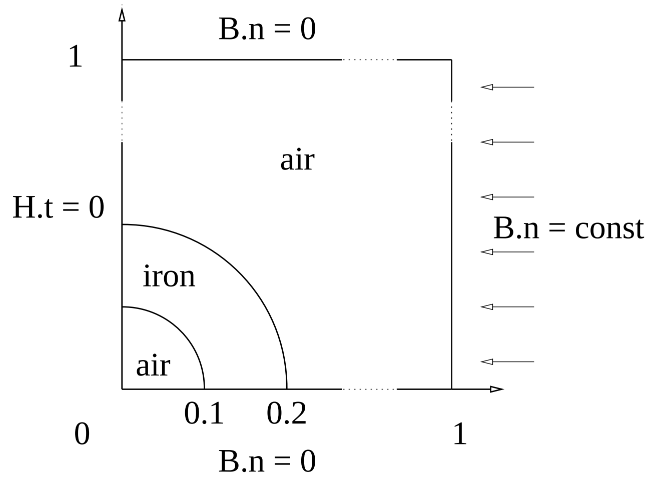
Sparsità

M e **A** sparse

- Stokes e Darcy
- Magnetostatica (Perugia, Simoncini, A. SISC 99) (Perugia, Simoncini 00)

$$\min_{\substack{\mathrm{div}\,\underline{B}=0\\ \mathrm{curl}\,\underline{H}=\underline{J}}} \frac{1}{2}\|\underline{B}-\mu\underline{H}\|^2 \qquad \text{in} \qquad \Omega$$

$$\text{with} \quad \underline{B}\cdot\underline{n}=g_1 \;\text{ on }\; \Gamma_B \qquad\qquad \underline{H}\wedge\underline{n}=\underline{g}_2 \;\text{ on }\; \Gamma_H.$$



Sparsità

$\mathbf{M}$ e $\mathbf{A}$ sparse

Sparsità

M e **A** sparse

- Sistema Lagrangiano risolto con MA57 (HSL) usando LDL^T

Sparsità

$\mathbf{M}$ e $\mathbf{A}$ sparse

- Sistema Lagrangiano risolto con MA57 (HSL) usando LDL^T

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{b} \end{bmatrix}.$$

Sparsità

$\mathbf{M}$ e $\mathbf{A}$ sparse

- Sistema Lagrangiano risolto con MA57 (HSL) usando LDL^T
- Fattorizzare $\mathbf{A}$ con MA49 (o MA48) ma $\tilde{\mathbf{M}}$ densa

Sparsità

M e **A** sparse

- Sistema Lagrangiano risolto con MA57 (HSL) usando LDL^T
- Fattorizzare **A** con MA49 (o MA48) ma $\tilde{\mathbf{M}}$ densa risolvere

$$\tilde{\mathbf{M}}\mathbf{z}_2 = \mathbf{h}_2 - \tilde{\mathbf{M}}_{12}^T\mathbf{z}_1$$

con CG più stopping criterium
H densa ma i vettori di Householder sparsi

Sparsità

M e **A** sparse

- Sistema Lagrangiano risolto con MA57 (HSL) usando LDL^T
- Fattorizzare **A** con MA49 (o MA48) ma $\tilde{\mathbf{M}}$ densa risolvere

$$\tilde{\mathbf{M}}\mathbf{z}_2 = \mathbf{h}_2 - \tilde{\mathbf{M}}_{12}^T \mathbf{z}_1$$

con CG più stopping criterium
H densa ma i vettori di Householder sparsi

$$\tilde{\mathbf{M}}_{ij} z = E_i^T H^T (M(HE_j z)), \text{ with } \quad (i,j) \;=\; (1,2).$$

$$E_1 = \begin{bmatrix} I_m \\ 0_{n-m,m} \end{bmatrix}, \text{ and } \quad E_2 = \begin{bmatrix} 0_{m,n-m} \\ I_{n-m} \end{bmatrix}.$$

Sparsità

IF $\quad ||\tilde{\mathbf{M}}_{22}\tilde{\mathbf{z}}_2 - fl(\mathbf{h}_2 - \tilde{\mathbf{M}}_{12}^T\mathbf{z}_1)||_2 \leq \eta||\tilde{\mathbf{M}}_{22}||_2||\tilde{\mathbf{z}}_2||_2 \quad$ THEN STOP ,

$\eta < 1$ fissata a-priori dall'utente

Sparsità

IF $||\tilde{\mathbf{M}}_{22}\tilde{\mathbf{z}}_2 - fl(\mathbf{h}_2 - \tilde{\mathbf{M}}_{12}^T\mathbf{z}_1)||_2 \leq \eta||\tilde{\mathbf{M}}_{22}||_2||\tilde{\mathbf{z}}_2||_2$ THEN STOP ,

$\eta < 1$ fissata a-priori dall'utente

$$(\tilde{\mathbf{M}}_{22} + \mathbf{E}_{22})\tilde{\mathbf{z}}_2 = fl(\mathbf{h}_2 - \tilde{\mathbf{M}}_{12}^T\mathbf{z}_1), \quad ||\mathbf{E}_{22}||_2 \leq \eta||\tilde{\mathbf{M}}_{22}||_2.$$

QP sparso

Siano $\bar{\mathbf{u}}$ e $\bar{\mathbf{x}}$ i valori calcolati delle soluzioni $\mathbf{u}$ and $\mathbf{x}$, con il **Null Space Algorithm** e il CG.
Se $n\varepsilon \ll 1$ esistono le matrici $\delta\mathbf{M}_1 \in \mathbf{R}^{n\times n}$ $\delta\mathbf{A}_1,\ \delta\mathbf{A}_2 \in \mathbf{R}^{n\times m}$, e il vettore $\delta\mathbf{q} \in \mathbf{R}^n$ tali che

$$\begin{bmatrix} \mathbf{M}+\delta\mathbf{M}_1+\delta\mathbf{M}_2 & \mathbf{A}+\delta\mathbf{A}_1 \\ (\mathbf{A}+\delta\mathbf{A}_2)^T & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} (\mathbf{q}+\delta\mathbf{q}) \\ \mathbf{b} \end{bmatrix}.$$

$$\begin{aligned} ||\delta\mathbf{A}_1||_F &\le c_2 m\varepsilon||\mathbf{A}||_F + \mathcal{O}(\varepsilon^2), \\ ||\delta\mathbf{A}_2||_F &\le c_2 m\varepsilon||\mathbf{A}||_F + \mathcal{O}(\varepsilon^2), \\ ||\delta\mathbf{q}||_2 &\le c_1\varepsilon||\mathbf{q}||_2 + \mathcal{O}(\varepsilon^2), \end{aligned}$$

e

$$\begin{aligned} ||\delta\mathbf{M}_1||_F &\le c_3(n,m)\varepsilon\|\mathbf{M}\|_2 + \mathcal{O}(\varepsilon^2), \\ ||\delta\mathbf{M}_2||_F &\le \eta(||\tilde{\mathbf{M}}_{22}||_2 + \mathcal{O}(\varepsilon))\ . \end{aligned}$$

Prospettive future

- Metodi Iterativi

Prospettive future

- Metodi Iterativi
  - precondizionatori basati su P

Prospettive future

- Metodi Iterativi
  - precondizionatori basati su $\mathbf{P}$ (Simoncini)

Prospettive future

- Metodi Iterativi
  - precondizionatori basati su $\mathbf{P}$ (Simoncini)
  - algoritmi alla $LSQR$ e $CRAIG$ (Benbow SIMAX 99), (Saunders BIT 95)

Prospettive future

- Metodi Iterativi
  - precondizionatori basati su $\mathbf{P}$ (Simoncini)
  - algoritmi alla $LSQR$ e $CRAIG$ (Benbow SIMAX 99), (Saunders BIT 95)
- Accuratezza dei metodi diretti con pivot statico

THE END ?