CNAc: Continuous Optimization
Problem set 5 — interior-point methods

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Instructions: Asterisked problems are intended as a homework assignment. Please put your solutions in Denis Zuev's pigeon hole at the Maths Institute by 9AM on Monday of 8th week.

A positive scalar sequence $\{\sigma_k\}$ with limit 0 is said to converge at a *Q-rate* q if

$$\lim_{k\to\infty} \frac{\sigma_{k+1}}{\sigma_k^q} \leq \kappa$$

for some constant κ—here "Q" stands for "Quotient", and the number q is sometimes known as the *Q-factor*. The convergence is said to be *Q-linear* if $q = 1$ and $\kappa < 1$, it is *Q-superlinear* if $q > 1$ or $q = 1$ and $\kappa = 0$ and *Q-quadratic* if $q = 2$. The Q-rate of convergence a vector sequence $\{x_k\}$ to its limit x_* is that of the sequence $\{\sigma_k\}$ where $\sigma_k = \|x_k - x_*\|$ for some appropriate norm.

***Problem 1.**
What is the Q-rate of convergence of the following sequences $\{\sigma_k\}$?

(a) $\sigma_k = 1/\log(k+1)$

(b) $\sigma_k = 2^{-k}$

(c) $\sigma_k = 2^{-k^2}$

(d) $\sigma_k = 2^{-2^k}$

***Problem 2.**
Consider the *reciprocal* barrier function

$$\Phi(x,\mu) = f(x) + \sum_{i=1}^{m} \frac{\mu}{c_i(x)}$$

for the inequality constrained optimization problem of minimizing $f(x)$ subject $c_i(x) \geq 0$ for $i = 1, \ldots, m$. By setting the gradient of Φ to zero, suggest suitable Lagrange multiplier estimates $y(x)$. Hence state and prove the analog of Theorem 6.1 for the reciprocal barrier function.

***Problem 3.**

(a) Show that the logarithmic barrier function for the problem of minimizing $1/(1+x^2)$ subject to $x \geq 1$ is unbounded from below for all μ.
[*Thus the barrier function approach will not always work.*]

(b) Find the minimizer $x(\mu)$, and its related Lagrange multiplier estimate $y(\mu)$, of the logarithmic barrier function for the problem of minimizing $\frac{1}{2}x^2$ subject to $x \geq 2a$ where $a > 0$. What is the rate of convergence of $x(\mu)$ to x_* as a function of μ? And the rate of convergence of $y(\mu)$ to y_* as a function of μ?
[*Problems with strictly complementary solutions generally have* $x(\mu) - x_* = O(\mu)$ *and* $y(x(\mu)) - y_* = O(\mu)$ *as* $\mu \to 0$.]

(c) Find the minimizer $x(\mu)$, and its related Lagrange multiplier estimate $y(\mu)$, of the logarithmic barrier function for the problem of minimizing $\frac{1}{2}x^2$ subject to $x \geq 0$. How do the errors $x(\mu) - x_*$ and $y(\mu) - y_*$ behave as a function of μ?
[*Without strict complementarity, the errors* $x(\mu) - x_*$ *and* $y(x(\mu)) - y_*$ *are generally larger than in the strictly complementary case.*]