Finding a point in the relative interior of a polyhedron, with applications to compressed sensing

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***Abstract*—Consider a system of finitely many equalities and inequalities that depend linearly on N variables. We propose an algorithm that provably finds a point in the relative interior of the polyhedron described by these constraints, thus allowing the identification of the affine dimension of this set (often smaller than N). It can therefore be employed to find a starting point for the class of interior-point methods for linear programming, and also as a *preprocessor* for the latter problem class that removes superfluous or implied constraints, with strong guarantees of convergence. In particular, it may be used to solve the feasibility problems that occur in sparse approximation when prior information is included (Donoho & Tanner, 2008).**

I. Overview of results

A new initialization or 'Phase I' strategy for feasible interior point methods for linear programming is proposed that computes a point on the primal-dual central path associated with the linear program. Provided there exist primal-dual strictly feasible points — an all-pervasive assumption in interior point method theory that implies the existence of the central path — our initial method (Algorithm 1) is globally Q-linearly and asymptotically Q-quadratically convergent, with a provable worst-case iteration complexity bound. When this assumption is not met, the numerical behaviour of Algorithm 1 is highly disappointing, even when the problem is primal-dual feasible. This is due to the presence of implicit equalities, inequality constraints that hold as equalities at all the feasible points. Controlled perturbations of the inequality constraints of the primal-dual problems are introduced — geometrically equivalent to enlarging the primal-dual feasible region and then systematically contracting it back to its initial shape — in order for the perturbed problems to satisfy the assumption. Thus Algorithm 1 can successfully be employed to solve each of the perturbed problems. We show that, when there exist primal-dual strictly feasible points of the original problems, the resulting method, Algorithm 2, finds such a point in a finite number of changes to the perturbation parameters. When implicit equalities are present, but the original problem and its dual are feasible, Algorithm 2 asymptotically detects *all* the primal-dual implicit equalities and generates a point in the relative interior of the primal-dual feasible set. Algorithm 2 can also asymptotically detect primal-dual infeasibility. Successful numerical experience with Algorithm 2 on linear programs from the standard test sets NETLIB and CUTEr [3], both with and without any significant preprocessing of the problems, indicates that Algorithm 2 may be used as an *algorithmic preprocessor* for removing implicit equalities, with theoretical guarantees of convergence.

The work to be presented is described at length in [1], where extensive analysis and numerical experiments with Algorithm 2 are given. and Algorithm 2 has been rigurously implemented in Fortran 90 under the name WCP (Well-Centred Point), and is part of the principal optimization library GALAHAD [4]; see also `http://galahad.rl.ac.uk`. WCP is freely available for academic use.

II. Application to compressed sensing

The proposed algorithm (Algorithm 2 in [1]) provably identifies all implicit equalities, namely, all the inequalities that in fact hold as equalities at all the feasible points of the set determined by the constraints; in other words, it determines the affine dimension of the constrained set. For most existing software, this type of algebraic and geometric dependencies is dealt with in a *preprocessing* phase of the solution process, which incorporates highly heuristical techniques for degeneracy removal. Thus our method stands out among such techniques, due to its strong theoretical guarantees of convergence. Furthermore, its powerful detection properties, namely the fact that it can determine a 'maximally non-sparse' solution of a set of linear constraints, make our algorithm ideally suited for the compressed sensing applications described in [2]. In particular, in [2], the question of recoverability of a k-sparse solution, with nonnegative entries, of an underdetermined linear system of equations is addressed; it is shown that for numerous random matrix ensembles, with high probability, if there is such a sparse solution, then it is the unique point of the intersection of the nonnegative orthant with the given linear system of equations. Thus in this application, one needs to be able to find out if a given system of linear equalities and inequalities has a unique (unknown) solution. In particular, one must solve the following problem: consider a vector $x \in \mathbb{R}^N_+ \geq 0$ with k nonzeros and a random matrix $A \in \mathbb{R}^{n\times N}$; then, does there exist another nonnegative vector $y \in \mathbb{R}^N_+ \geq 0$ such that $Ay = Ax$ (where x is not known a priori)? Assume we apply our algorithm to this feasibility problem and recover an m-sparse vector y_0 with $m \leq n$; then, by the construction of the algorithm, y_0 is the 'least sparse' feasible solution that there is, and its support set contains the support set of x. Thus $y_0 - x$ is at most m sparse and belongs to the null space of A. For these kinds of applications, however, A is commonly assumed to be in general position, and since $m \leq n$, we must have $y_0 = x$ and so we have recovered the sparsest solution and there is no other feasible point of $\{y : \ Ay = Ax, \, y \geq 0\}$. If the point the algorithm finds is $m-$sparse with $m > n$, then it is not unique. Thus we can use our algorithm to fully answer the feasibility question that arises in the above application. Similarly, in [2], the case of the intersection of a subspace with the hypercube is addressed, and a similar feasibility problem ensues, that our algorithm can fully solve.

References

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