A FILTER METHOD WITH UNIFIED STEP COMPUTATION FOR NONLINEAR OPTIMIZATION*

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Abstract. We present a filter line search method for solving general nonlinear and nonconvex optimization problems. The method is of the filter variety but uses a robust (always feasible) subproblem based on an exact penalty function to compute a search direction. This contrasts traditional filter methods that use a (separate) restoration phase designed to reduce infeasibility until a feasible subproblem is obtained. Therefore, an advantage of our approach is that every trial step is computed from subproblems that value reducing both the constraint violation and the objective function. Moreover, our step computation involves subproblems that are computationally tractable and utilize second derivative information when it is available. The formulation of each subproblem and the choice of weighting parameter is crucial for obtaining an efficient, robust, and practical method. Our strategy is based on steering methods designed for exact penalty functions but is fortified with a trial step convexification scheme that ensures that a single quadratic optimization problem is solved per iteration. Moreover, we use local feasibility estimates that emerge during the steering process to define a new and improved margin (envelope) of the filter. Under common assumptions, we show that the iterates converge to a local first-order solution of the optimization problem from an arbitrary starting point.

Key words. filter, restoration phase, large-scale, sequential quadratic programming, nonlinear programming

AMS subject classifications. 49M05, 49M15, 65K05, 65K10, 65K15

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1. Introduction. This paper considers the general nonlinear optimization problem

\[
\minimize_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad c(x) \geq 0,
\]

where both the objective function \( f : \mathbb{R}^n \to \mathbb{R} \) and the constraint function \( c : \mathbb{R}^n \to \mathbb{R}^m \) are assumed to be twice continuously differentiable. We seek a first-order KKT point \((x, y)\) that satisfies

\[
F_{\text{KKT}}(x, y) := \begin{pmatrix} g(x) - J(x)^T y \\ \min \[ c(x), y \] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

where \( g(x) := \nabla f(x) \in \mathbb{R}^n \) is the gradient of the objective function, \( J(x) := \nabla c(x) \in \mathbb{R}^{m \times n} \) is the Jacobian of the constraint function, \( y \) is the Lagrange multiplier vector, and the minimum is taken componentwise. Our algorithm may easily handle constraints with general lower/upper bounds and handle equality constraints directly, i.e., it does not replace them with pairs of inequality constraints. Problems of this type arise naturally in many areas, including optimal control [2, 3, 6, 25, 32], resource

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allocation [1, 31], solution of equilibrium models [20, 38], and structural engineering [5, 34].

Popular methods for solving (1.1) can broadly be characterized as interior-point or active-set methods. Interior-point algorithms [44, 46, 47] offer polynomial-time complexity bounds in many cases and readily scale up to problems involving millions of variables. Their main disadvantage is the inability to use effectively a good initial estimate of a solution. In fact, many interior-point methods immediately move the initial guess into the strict interior of the feasible region. It is from this interior location that future iterates are forced to remain and it justifies the name “interior-point” methods; more modern “infeasible” interior-point methods avoid this defect to some degree.

Active-set methods [10, 16, 27, 28, 29, 30, 37, 39] complement interior-point methods since they naturally utilize information derived from a good estimate of a solution. In fact, if the optimal active set (the set containing those constraints satisfied as equalities at a solution) was known in advance, then problem (1.1) could be solved as an equality constrained problem and its combinatorial nature would be eliminated. It is precisely this property that makes active-set methods widely used to solve the previously mentioned class of problems. The main weakness of active-set algorithms is that each subproblem typically requires the solution of a linear or quadratic program, which is often expensive when compared to interior-point methods that require a single linear system solve per iteration.

In this paper we describe an active-set method that generates a sequence of iterates from the solutions of subproblems defined by local models of the nonlinear problem functions. The subproblems are always feasible since they are based on an exact penalty function. To ensure that these models result in productive steps, we use steering techniques [11] to adaptively adjust the weighting (penalty) parameter. In contrast to original steering methods, we use a step convexification procedure similar to [7, 18] to avoid solving multiple quadratic programs during each iteration.

To provide convergence guarantees, we must include a mechanism for determining when one point is “better” than another. A merit (penalty) function or a filter is among the most common tools used for this purpose. A merit function combines the objective function and a measure of constraint violation into a single function, whereby their individual perceived importance is determined by a weighting parameter. The quality of competing points is then measured by comparing their respective merit function values. A potential weakness is that the quality of iterates depends on the value of the weighting parameter, which can make step acceptance sensitive to its value. In part, filter methods surfaced to mitigate this parameter dependence. In the context of nonlinear optimization, filter methods were introduced by Fletcher, Leyffer, and Toint [22, 23] and have since been rather popular [13, 14, 15, 21, 24, 44]. A filter views problem (1.1) as a multicriteria optimization problem consisting of minimizing the objective function and minimizing some measure of the constraint violation, with certain preference given to the latter. Roughly, a trial iterate is then considered acceptable if it has a smaller value of either the objective function or the constraint violation compared to the previously encountered points. Consequently, it is often the case that filter methods accept more iterates and perform better. It should be mentioned, however, that every known provably convergent filter method has a weak dependence between these two criteria that is embedded in the step acceptance criteria. In fact, this observation partly motivated the work on flexible penalty methods by Curtis and Nocedal [19]. They describe how a single element filter is essentially equivalent to the union of points acceptable to the $\ell_1$-penalty function defined over an interval of weighting parameter values.
A great disadvantage of filter methods is that they (traditionally) require the use of a restoration phase. A restoration phase is (typically) entered when the subproblem used to compute trial steps is infeasible; some algorithms, e.g., [46], enter the restoration phase for additional reasons. When this situation occurs, the restoration phase is triggered and a sequence of iterates focused on reducing the constraint violation is computed until the desired subproblem becomes feasible. During this phase, the objective function is essentially ignored, which is highly undesirable from both a practical and a computational perspective.

Our active-set method is globalized by using a filter but never needs to enter a (traditional) restoration phase. This is accomplished by using subproblems that are always feasible and, in certain instances, allow for the acceptance of iterates that decrease both the exact penalty function and the constraint violation. In essence, we replace an undesirable restoration phase with an attractive penalty phase. Thus, we combine ideas from both filter and penalty methods to formulate a robust and effective method; we believe this further builds upon the basic observations in [19].

This paper contains three main contributions. First, we present a filter method that avoids a traditional and highly undesirable restoration phase. To this end, we utilize subproblems based on exact penalty functions that are always feasible and formed from models of both the objective function and constraint violation. Second, our method incorporates second derivative information without requiring global minimizers of nonconvex constrained subproblems (cf. [21]). Our step computation is most similar to [28, 30], which was described in the context of line search and trust-region penalty methods. Third, we use local feasibility estimates that emerge during the steering step computation to define a new and improved margin (envelope) of the filter. This allows us to define an adaptive and practical margin.

Our work is not the only method designed to resolve weaknesses in traditional filter methods. Chen and Goldfarb [12] presented an interior point method that uses two penalty functions to determine step acceptance: a piecewise linear penalty function whose break points are essentially elements in the filter, and the $\ell_2$-penalty function. Under this scheme, a trial step is accepted if it provides sufficient reduction for either penalty function.

The remainder of this paper is organized as follows. In section 2 we describe the algorithm in detail, and in section 3 we prove that it is well-posed. We provide convergence results in section 4 and conclude with final remarks in section 5.

2. A filter sequential quadratic programming method. In this section we describe our new filter sequential quadratic programming method, FiSQP. The algorithm is iterative and relies on computing trial steps from carefully constructed subproblems. These subproblems and the resulting trial steps are explained in sections 2.1–2.6. In section 2.7 we introduce the filter construct and related terminology; we emphasize that acceptability to the filter is only a necessary condition for accepting a trial iterate. A full statement and description of the algorithm are given in section 2.8.

Our step computation is based on the $\ell_1$-penalty function

$$\phi(x; \sigma) := f(x) + \sigma v(x), \quad (2.1)$$

where $\sigma$ is a positive weighting parameter and the constraint violation at $x$ is defined by
and where the maximum is taken componentwise. We use linear and quadratic model approximations of the objective function $f$ given by

$$
\ell^f(s; x) := f(x) + g(x)^T s \quad \text{and} \quad q^f(s; x, M) := \ell^f(s; x) + \frac{1}{2}s^T Ms = f(x) + g(x)^T s + \frac{1}{2}s^T Ms
$$

for a given symmetric matrix $M \in \mathbb{R}^{n \times n}$ and use a piecewise-linear approximation to the constraint violation function $v$ given by

$$
\ell^v(s; x) := \|[c(x) + J(x)s]\|_1
$$

to form the following linear and quadratic models of $\phi$:

$$
\ell^\phi(s; x, \sigma) := \ell^f(s; x) + \sigma \ell^v(s; x) = f(x) + g(x)^T s + \sigma \|[c(x) + J(x)s]\|_1
$$

$$(2.4) \quad q^\phi(s; x, M, \sigma) := q^f(s; x, M) + \sigma \ell^v(s; x) = f(x) + g(x)^T s + \frac{1}{2}s^T Ms + \sigma \|[c(x) + J(x)s]\|_1.
$$

Using these models we may predict the change in $v$ with the function

$$
\Delta \ell^v(s; x) := \ell^v(0; x) - \ell^v(s; x) = \|[c(x)]\|_1 - \|[c(x) + J(x)s]\|_1,
$$

the change in $f$ with the functions

$$
\Delta \ell^f(s; x) := \ell^f(0; x) - \ell^f(s; x) = -g(x)^T s \quad \text{and}
$$

$$
\Delta q^f(s; x, M) := q^f(0; x, M) - q^f(s; x, M) = \Delta \ell^f(s; x) - \frac{1}{2}s^T Ms = -g(x)^T s - \frac{1}{2}s^T Ms,
$$

and the change in the penalty function $\phi$ with the functions

$$
\Delta \ell^\phi(s; x, \sigma) := \ell^\phi(0; x, \sigma) - \ell^\phi(s; x, \sigma) = \Delta \ell^f(s; x) + \sigma \Delta \ell^v(s; x)
$$

$$
= -g(x)^T s + \sigma \left( \|[c(x)]\|_1 - \|[c(x) + J(x)s]\|_1 \right)
$$

and

$$
\Delta q^\phi(s; x, M, \sigma) := q^\phi(0; x, M, \sigma) - q^\phi(s; x, M, \sigma) = \Delta \ell^\phi(s; x, \sigma) - \frac{1}{2}s^T Ms
$$

$$
= -g(x)^T s - \frac{1}{2}s^T Ms + \sigma \left( \|[c(x)]\|_1 - \|[c(x) + J(x)s]\|_1 \right).
$$

For the remainder of this section, let $(x_k, y_k)$ denote the current estimate of a solution to (1.1).

2.1. The steering step $s_k^p$. In order to strike a proper balance between reducing the objective function and the constraint violation, we compute a steering step $s_k^p$ as a solution to the linear program

$$
(2.10) \quad \text{minimize} \quad c^T r \quad \text{subject to} \quad c_k + J_k s + r \geq 0, \quad r \geq 0, \quad \|s\|_\infty \leq \delta_k,
$$

where $c_k = c(x_k)$, $J_k = J(x_k)$, $\delta_k \in [\delta_{\min}, \delta_{\max}]$, and $0 < \delta_{\min} \leq \delta_{\max} < \infty$. Problem (2.10) is equivalent to the nonsmooth problem

$$
(2.11) \quad \text{minimize} \quad \ell^v(s; x_k) \quad \text{subject to} \quad \|s\|_\infty \leq \delta_k
$$
since \( s \) solves (2.11) if and only if \((s, r)\) solves (2.10), where \( r = \max(-c_k + J_k s, 0) \).

Since \( \ell^v(0; x_k) = v(x_k) \), \( \ell^v \) is a convex function, and \( s = 0 \) is feasible for (2.11), it follows from (2.6) that \( \Delta \ell^v(s^*; x_k) \geq 0 \). The quantity \( \Delta \ell^v(s^*; x_k) \) is the best local improvement in linearized constraint feasibility for steps of size \( \delta_k \).

All methods for nonconvex optimization may converge to an infeasible point that is a local minimizer of the constraint violation as measured by \( v \). Points of this type are known as infeasible stationary points, which we now define by utilizing the steering subproblem.

**Definition 2.1** (infeasible stationary point). The vector \( x^i \) is an infeasible stationary point if \( v(x^i) > 0 \) and \( \Delta \ell^v(s^i; x^i) = 0 \), where \( s^i = \arg\min_{s \in \mathbb{R}^n} \ell^v(s; x^i) \) subject to \( \|s\|_\infty \leq \delta \) for some \( \delta > 0 \).

**2.2. The predictor step** \( s^P_k \). The predictor step is computed as the unique solution to one of the following strictly convex minimization problems:

\[
\begin{align*}
(2.12a) \quad & \arg\min_{s \in \mathbb{R}^n} f_k + g_k^T s + \frac{1}{2} s^T B_k s \quad \text{subject to} \quad c_k + J_k s \geq 0 \\
(2.12b) \quad & \arg\min_{s \in \mathbb{R}^n} q^0(s; x_k, B_k, \sigma_k), \quad \text{otherwise},
\end{align*}
\]

where \( \sigma_k > 0 \) is the \( k \)th value of the penalty parameter, \( f_k = f(x_k), g_k = \nabla f(x_k), c_k = c(x_k), J_k = \nabla c(x_k) \), \( B_k \) is a positive-definite matrix that we are free to choose such that \( B_k \approx \nabla^2 L(x_k, y_k) \), and the Lagrangian \( L \) is defined by \( L(x, y) = f(x) - c(x)^T y \).

Analogous to the steering subproblem, the nonsmooth minimization problem (2.12b) is equivalent to the smooth problem

\[
(2.13) \quad \min_{s \in \mathbb{R}^n, r \in \mathbb{R}^n} f_k + g_k^T s + \frac{1}{2} s^T B_k s + \sigma_k r^T r \quad \text{subject to} \quad c_k + J_k s + r \geq 0, \quad r \geq 0,
\]

which is the problem solved in practice. We use \( y^P_k \) to denote the Lagrange multiplier vector for the constraint \( c_k + J_k s \geq 0 \) in (2.12a) and \( c_k + J_k s + r \geq 0 \) in (2.13) (equivalently (2.12b)). Possible choices for the positive-definite matrix include (i) the trivial diagonal matrix based on the Barzilai-Borwein [4] method, (ii) quasi-Newton updates such as BFGS [40] and L-BFGS [33], and (iv) modified Cholesky factorizations [26, 43]. Moreover, relaxing the positive-definition assumption on \( B_k \) may also be possible provided conditions such as those used in [18] are enforced to ensure that sufficient descent directions are computed. Since this relaxation would introduce unnecessary complications into our algorithm and require the use of an indefinite QP solver, we will assume throughout that \( B_k \) is a positive-definite matrix.

The next result shows how convergence to KKT points may be deduced from the predictor problem.

**Lemma 2.2.** Suppose that \( x_* \) satisfies

\[
\begin{align*}
& v(x_*) = 0 \quad \text{and} \\
& 0 = \arg\min_{s \in \mathbb{R}^n} f(x_*) + g(x_*)^T s + \frac{1}{2} s^T B s \quad \text{subject to} \quad c(x_*) + J(x_*) s \geq 0
\end{align*}
\]

for some positive definite matrix \( B \), and let \( y_* \) denote the associated Lagrange multiplier vector. Then, it follows that \((x_*, y_*)\) is a KKT point for problem (1.1) as defined by (1.2).
Proof. Since $B$ is positive definite, $s = 0$ is the unique solution to the optimization problem in (2.14). It then follows from the first-order necessary optimality conditions at $s = 0$ that
\[ g(x_*) = J(x_*)^T y_* \quad \text{and} \quad \min (c(x_*), y_*) = 0, \]
where $y_*$ is the Lagrange multiplier for the constraint $c(x_*) + J(x_*) s \geq 0$. It now follows from Definition 1.2 that $(x_*, y_*)$ is a KKT point for problem (1.1).

2.3. The search direction $s_k$. The steering direction $s_k^*$ provides a measure of local progress in infeasibility. Since we desire a search direction $s_k$ that makes progress toward feasibility, we define
\[ s_k := (1 - \tau_k) s_k^* + \tau_k s_k^p, \]
where $\tau_k$ is the largest number on $[0, 1]$ such that
\[ \Delta \ell^v(s_k; x_k) \geq \eta_k \Delta \ell^v(s_k^p; x_k) \geq 0 \quad \text{for some} \quad \eta_k \in (0, 1). \]

The direction $s_k$ is not an infeasible stationary point as given by Definition 2.1, then $\tau_k > 0$.

Proof. If $v(x_k) = 0$, then $\Delta \ell^v(s_k^*; x_k) = 0$. It then follows from (2.12a) that $c_k + J_k s_k^p \geq 0$, which in turn implies that $\Delta \ell^v(s_k^p; x_k) = 0$. Thus, the choice $\tau_k = 1$ satisfies (2.15) and (2.16).

Now suppose that $v(x_k) > 0$ and define
\[ s(\tau) = (1 - \tau)s_k^* + \tau s_k^p \]
so that $\lim_{\tau \downarrow 0} s(\tau) = s_k^*$. It then follows from continuity of $\Delta \ell^v(\cdot; x_k)$ and the fact that $\Delta \ell^v(s_k^*; x_k) > 0$ since $x_k$ is not an infeasible stationary point by assumption that
\[ \lim_{\tau \downarrow 0} \Delta \ell^v(s(\tau); x_k) = \Delta \ell^v(s_k^*; x_k) > 0. \]

Therefore, there exists $\tau' > 0$ such that
\[ |\Delta \ell^v(s(\tau); x_k) - \Delta \ell^v(s_k^*; x_k)| < (1 - \eta_0) \Delta \ell^v(s_k^*; x_k) \quad \text{for all} \quad \tau \in [0, \tau'] \]
since $\eta_0 \in (0, 1)$ in (2.16) and $\Delta \ell^v(s_k^*; x_k) > 0$. However, this implies that
\[ \Delta \ell^v(s(\tau); x_k) \geq \eta_0 \Delta \ell^v(s_k^*; x_k) \quad \text{for all} \quad \tau \in [0, \tau'], \]
which guarantees that $t_k \geq \tau' > 0$.

We now proceed to show that if $\Delta \ell^v(s_k^*; x_k) > 0$, then $s_k$ is a descent direction for $v(\cdot)$. We require the definition of the directional derivative of a function.

Definition 2.4. The directional derivative of a function $h(\cdot)$ in the direction $d$ and at the point $x$ is defined (when it exists) as
\[ [D_d h](x) := \lim_{t \downarrow 0} \frac{h(x + td) - h(x)}{t}. \]
We now show that the directional derivative is bounded by the negative of the change in its model.

**Lemma 2.5.** At any point \( x \) and for any direction \( d \), it follows that

\[
[D_d v](x) \leq -\Delta \ell^v(d; x),
\]

where the function \( D_d v \) is the directional derivative of \( v \) in the direction \( d \).

**Proof.** Since \( \ell^v \) is a convex function and \( \ell^v(0; x) \) is finite, it follows from [42, Theorem 23.1] that

\[
\frac{\ell^v(td; x) - \ell^v(0; x)}{t} \leq -\Delta \ell^v(d; x),
\]

is monotonically nondecreasing with \( t \), \( [D_d \ell^v](0; x) \) exists, and

\[
[D_d \ell^v](0; x) = \inf_{t > 0} \frac{\ell^v(td; x) - \ell^v(0; x)}{t}.
\]

It then follows from [9, Lemma 3.1], (2.17), and the definition of \( \Delta \ell^v \) that

\[
[D_d v](x) = [D_d \ell^v](0; x) \leq \ell^v(d; x) - \ell^v(0; x) = -\Delta \ell^v(d; x),
\]

which implies that \( s_k \) is a descent direction for \( v \) when our infeasibility measure is positive.

**Lemma 2.6.** If \( \Delta \ell^v(s_k^r; x_k) > 0 \), the direction \( s_k \) is a descent direction for \( v \) at the point \( x_k \), i.e.,

\[
[D_{s_k} v](x_k) \leq -\Delta \ell^v(s_k; x_k) \leq -\eta_c \Delta \ell^v(s_k^r; x_k) < 0,
\]

where \( \eta_c \) is defined in (2.16).

**Proof.** It follows directly from Lemma 2.5, (2.16), and \( \Delta \ell^v(s_k^r; x_k) > 0 \) that

\[
[D_{s_k} v](x_k) \leq -\Delta \ell^v(s_k; x_k) \leq -\eta_c \Delta \ell^v(s_k^r; x_k) < 0,
\]

which implies that \( s_k \) is a descent direction for \( v \) at the point \( x_k \).

We now consider the case when our infeasibility measure is zero.

**Lemma 2.7.** If \( \Delta \ell^v(s_k^r; x_k) = 0 \), then one of the following must occur:

(i) \( v(x_k) > 0 \) and \( x_k \) is an infeasible stationary point or

(ii) \( v(x_k) = 0 \) and \( \Delta \ell^\phi(s_k; x_k, \sigma) \geq \frac{1}{2} s_k^p B_k s_k^p \) for all \( 0 < \sigma < \infty \).

**Proof.** If \( v(x_k) > 0 \), then by Definition 2.1, \( x_k \) is an infeasible stationary point which is part (i). Now, suppose that \( v(x_k) = 0 \). As in the proof of Lemma 2.3, it follows that

\[
\Delta \ell^\phi(s_k^p; x_k) = 0, \quad \tau_k = 1, \quad \text{and} \quad s_k = s_k^p.
\]

We may then use the definition of \( s_k^r \) in (2.12a), (2.18), and (2.7b) to conclude that

\[
0 \leq \Delta q^\phi(s_k^r; x_k, B_k, \sigma_k) = \Delta q^f(s_k^r; x_k, B_k) = \Delta \ell^f(s_k^r; x_k) - \frac{1}{2} s_k^p B_k s_k^p,
\]

which yields \( \Delta \ell^f(s_k^p; x_k) \geq \frac{1}{2} s_k^p B_k s_k^p \). Combining this with (2.8) and (2.18), we have that

\[
\Delta \ell^\phi(s_k; x_k, \sigma) = \Delta \ell^f(s_k; x_k) + \sigma \Delta \ell^v(s_k; x_k)
\]

\[= \Delta \ell^f(s_k; x_k) = \Delta \ell^f(s_k^r; x_k) \geq \frac{1}{2} s_k^p B_k s_k^p \text{ for all } \sigma,
\]

which completes the proof.

\[\square\]
2.4. Updating the weighting parameter. By design, the trial step $s_k$ is a descent direction for $v$ when local improvement in feasibility is possible. Since the weighting parameter provides a balance between reducing the objective function and the constraint violation, it makes sense to adjust the weighting parameter so that $s_k$ is also a descent direction for $\phi$. This is accomplished by defining

$$
\sigma_{k+1} = \begin{cases} 
\sigma_k & \text{if } \Delta \ell^\phi(s_k; x_k, \sigma_k) \geq \sigma_k \eta_v \Delta \ell^v(s_k^*; x_k), \\
\max \left\{ \sigma_k, \sigma_{\text{inc}}, \frac{-\Delta \ell^\phi(s_k; x_k)}{\Delta \ell^v(s_k; x_k) - \eta_v \Delta \ell^v(s_k^*; x_k)} \right\} & \text{otherwise}
\end{cases}
$$

for some $\sigma_{\text{inc}} > 0$ and $\eta_v$ satisfying $0 < \eta_v < \eta_v < 1$, where $\eta_v$ is defined in (2.16).

**Lemma 2.8.** If $x_k$ is not an infeasible stationary point, then the parameter update (2.19) is well defined and ensures that

$$
\Delta \ell^\phi(s_k; x_k, \sigma_{k+1}) \geq \sigma_{k+1} \eta_v \Delta \ell^v(s_k^*; x_k) \geq 0 \quad \text{for all } k \geq 0.
$$

**Proof.** If $\Delta \ell^\phi(s_k; x_k, \sigma_k) \geq \sigma_k \eta_v \Delta \ell^v(s_k^*; x_k)$, then the desired result immediately follows from the update $\sigma_{k+1} = \sigma_k$. Thus, for the remainder of the proof we assume that

$$
\Delta \ell^\phi(s_k; x_k, \sigma_k) < \sigma_k \eta_v \Delta \ell^v(s_k^*; x_k).
$$

Suppose, for a contradiction, that $\Delta \ell^v(s_k^*; x_k) = 0$. Since $x_k$ is not an infeasible stationary point by assumption, it follows that $v(x_k) = 0$. Then, it follows from Lemma 2.7 and the fact that $B_k$ is positive definite by assumption that $\Delta \ell^\phi(s_k; x_k, \sigma_k) \geq \frac{1}{2} s_k^T B_k s_k^* \geq 0$, which contradicts (2.21) since $\Delta \ell^v(s_k^*; x_k) = 0$. Thus, we conclude that $\Delta \ell^v(s_k^*; x_k) > 0$. Combining this with the choice $0 < \eta_v < \eta_v < 1$ in (2.19) and (2.16) we conclude that $\Delta \ell^\phi(s_k; x_k) \geq \eta_v \Delta \ell^v(s_k^*; x_k) > \eta_v \Delta \ell^v(s_k^*; x_k) > 0$, and thus

$$
\eta_v \Delta \ell^v(s_k^*; x_k) - \Delta \ell^v(s_k^*; x_k) < 0.
$$

It then follows from (2.8), (2.21), (2.22), and the fact that $\sigma_k > 0$ that

$$
\Delta \ell^f(s_k; x_k) = \Delta \ell^\phi(s_k; x_k, \sigma_k) - \sigma_k \Delta \ell^v(s_k; x_k) < \sigma_k [\eta_v \Delta \ell^v(s_k^*; x_k) - \Delta \ell^v(s_k; x_k)] < 0.
$$

Inequalities (2.22) and (2.23) imply that the penalty parameter update (2.19) is well-defined and positive.

It now follows from (2.19) that

$$
\sigma_{k+1} \geq \frac{-\Delta \ell^\phi(s_k; x_k)}{\Delta \ell^v(s_k; x_k) - \eta_v \Delta \ell^v(s_k^*; x_k)},
$$

which may then be combined with (2.22) to yield

$$
\sigma_{k+1} \eta_v \Delta \ell^v(s_k^*; x_k) \leq \Delta \ell^f(s_k; x_k) + \sigma_{k+1} \Delta \ell^v(s_k; x_k) = \Delta \ell^\phi(s_k; x_k, \sigma_{k+1}),
$$

which is the desired result (2.20). \qed

The next result will allow us to show that $s_k$ is a descent direction for $\phi$ under certain assumptions.

**Lemma 2.9.** For any given value of the penalty parameter $\sigma$, point $x$, direction $d$, and positive-definite matrix $B$, it follows that

$$
|D_\sigma \phi|(x; \sigma) \leq -\Delta \ell^\phi(d; x, \sigma) \leq -\Delta q^\phi(d; x, B, \sigma).
$$
Proof. Linearity of the directional derivative, (2.7a), Lemma 2.5, (2.8), (2.9), and the fact that $B_k$ is positive definite by choice imply that

$$\begin{align*}
[D_d\phi](x;\sigma) &= [D_df](x) + \sigma[D_d\ell](x) = -g(x)^T d + \sigma[D_d\ell](d; x) \\
&= -\Delta\ell^\phi(d; x, \sigma) = -\Delta g^\phi(d; x, B, \sigma) - \frac{1}{2} d^T Bd \\
&
\end{align*}$$

which is the desired result.

In most situations, we may now show that $s_k$ is a descent direction for the penalty function.

**Lemma 2.10.** If $x_k$ is neither an infeasible stationary point nor a KKT point for problem (1.1), then the direction $s_k$ is a descent direction for $\phi(x; \sigma_{k+1})$ at the point $x_k$, i.e.,

$$[D_{s_k}\phi](x_k; \sigma_{k+1}) \leq -\Delta\ell^\phi(s_k; x_k, \sigma_{k+1}) < 0.$$

**Proof.** If $\Delta\ell^\phi(s_k^2; x_k) > 0$, then $x_k$ cannot be an infeasible stationary point, and it follows from Lemma 2.9, Lemma 2.8, and (2.20) that $[D_{s_k}\phi](x_k; \sigma_{k+1}) \leq -\Delta\ell^\phi(s_k^2; x_k, \sigma_{k+1}) < 0$, which is the desired result. Conversely, if $\Delta\ell^\phi(s_k^2; x_k) = 0$, then $v(x_k) = 0$ since $x_k$ is not an infeasible stationary point by assumption. It now follows from Lemma 2.9, $v(x_k) = 0$, Lemma 2.7, the fact that $B_k$ is positive definite, and $s_k^2 \neq 0$ since $x_k$ is not a KKT point for problem (1.1) by assumption (see Lemma 2.2), that $[D_{s_k}\phi](x_k; \sigma_{k+1}) \leq -\Delta\ell^\phi(s_k; x_k, \sigma_{k+1}) \leq -\frac{1}{2} s_k^2 B_k s_k^2 < 0$, which completes the proof.

**2.5. The accelerator step $s_k^p$.** To improve performance, we compute an additional “acceleration” step; here we consider a single (simple) possibility, but other variants may be used [28].

Under common assumptions, the predictor step $s_k^p$ will ultimately correctly identify those constraints that are active at a local solution of (1.1) [41]. A prediction based on $s_k^p$ is formulated by

$$(2.24) \quad A_k := \{i : [c_k + J_k s_k^p]_i = 0\}.$$

It is then natural to compute an *accelerator* step $s_k^p$ as the solution to

$$(2.25) \quad \min_{s \in \mathbb{R}^n} q^f(s_k^p + s; x_k, H_k) \quad \text{subject to} \quad [J_k s]_{A_k} = 0, \quad \|s\|_2 \leq \delta_k^p,$$

where $\delta_k^p > 0$ is the trust-region radius, $H_k$ is the exact second derivative of the Lagrangian $\nabla_{xx} L(x_k, y_k)$, and $y_k$ is a suitable Lagrange multiplier vector such as those from the predictor subproblem. (In fact, our global convergence analysis allows for any symmetric bounded sequence $\{H_k\}$, but here for concreteness we simply use $H_k = \nabla_{xx}^2 L(x_k, y_k)$.) We note that subproblem (2.25) may be solved, for example, with the projected GLTR algorithm. (See [17, section 7.5.4] and the notes at the end that describe how to cope with the affine constraints $[J_k s]_{A_k} = 0$. It can be shown that if $c_k + J_k s \geq 0$ is feasible, $\sigma_k$ is sufficiently large, and $x_k$ is “close enough” to a solution of (1.1) that satisfies certain second-order sufficient optimality conditions, then $s_k^p$ is the solution to

$$(2.26) \quad \min_{s \in \mathbb{R}^n} q^f(s; x_k, H_k) \quad \text{subject to} \quad c_k + J_k s \geq 0,$$

which is the traditional SQP subproblem. However, our method of step computation is robust, whereas the generally nonconvex subproblem (2.26) introduces many points of contention such as multiple solutions, unboundedness, and inconsistent constraints.
2.6. The Cauchy steps $s_k^{qf}$ and $s_k^{q\phi}$. Since the matrix $B_k$ is positive definite by construction and the exact second derivative matrix $H_k$ is generally an indefinite matrix, they may differ dramatically. To account for this when assessing overall step acceptance, we define and use a Cauchy-$f$ step $s_k^{qf}$ and a Cauchy-$\phi$ step $s_k^{q\phi}$ as follows.

Given the search direction $s_k$, we define the Cauchy-$f$ step as

$$s_k^{qf} := \alpha_k^f s_k, \quad \text{where } \alpha_k^f := \text{argmin}_{0 \leq \alpha \leq 1} q^f(\alpha s_k; x_k, H_k).$$

Similarly, we define the Cauchy-$\phi$ step as

$$s_k^{q\phi} := \alpha_k^\phi s_k, \quad \text{where } \alpha_k^\phi := \text{argmin}_{0 \leq \alpha \leq 1} q^\phi(\alpha s_k; x_k, H_k, \sigma_{k+1}).$$

The step size $\alpha_k^\phi$ may be found efficiently by examining the piecewise quadratic function $q^\phi(\alpha s_k; x_k, H_k, \sigma_{k+1})$ segment by segment between each derivative discontinuity.

2.7. The filter. The global convergence proof for our method is driven by maintaining/updating a filter $\mathcal{F}_k$ during each iteration. A filter is defined as follows, where $\mathbb{R}^+$ denotes the positive real numbers.

**Definition 2.11** (filter). A filter is any finite set of points in $\mathbb{R}^+ \times \mathbb{R}$.

The initial filter is defined to be $\mathcal{F}_0 = \emptyset$ and then sequentially updated in a manner that guarantees that $\mathcal{F}_k \subseteq \{(v_j, f_j) : 0 \leq j < k\}$. The decision to add certain ordered pairs to the filter depends on the concept of trial points being acceptable to the filter, which we now define.

**Definition 2.12** (acceptable to $\mathcal{F}_k$). We say that the point $x$ is acceptable to $\mathcal{F}_k$ if its associated ordered pair $(v(x), f(x))$ satisfies

$$v(x) \leq \max \left\{ v_i - \alpha_i \eta_i \Delta\ell^c(s_i^*; x_i), \beta v_i \right\} \quad \text{or} \quad f(x) \leq f_i - \gamma \min \left\{ v_i - \alpha_i \eta_i \Delta\ell^f(s_i^*; x_i), \beta v_i \right\}$$

for all $0 \leq i < k$ such that $(v_i, f_i) \in \mathcal{F}_k$ and some constants $\{\eta_i, \beta, \gamma\} \subset (0, 1)$.

The first inequality in (2.29) ensures that the constraint violation has been sufficiently reduced. We note that previous filter methods have not used the first quantity in the max on the right-hand side. Our improved condition takes advantage of the information supplied by the steering steps $s_k^*$. Previous filter methods may easily have requested a decrease in the constraint violation that was unreasonable. In these circumstances, the trust-region radius would be decreased until the subproblem became infeasible and then a feasibility restoration phase would be entered. Our modified definition provides a practical target constraint violation based on local information derived from the steering step $s_k^*$. The second inequality in (2.29) guarantees that the objective function is sufficiently smaller at the point $x$ than at points $x_i$ whose ordered pair is in the current filter $\mathcal{F}_k$. These two conditions provide a so-called margin around the elements of the filter.

Note that Definition 2.12 does not require and does not imply that the current vector $x_k$ is in $\mathcal{F}_k$ when determining acceptability. During our search for an improved estimate of a solution to (1.1), it often does not make sense to accept a new point unless it is acceptable to the current filter and better than the current point $x_k$. This leads to the following definition.

**Definition 2.13** (acceptable to $\mathcal{F}_k$ augmented by $x_k$). We say that $x$ is acceptable to $\mathcal{F}_k$ augmented by $x_k$ if $x$ is acceptable to $\mathcal{F}_k$ as given by Definition 2.12 and (2.29) holds with $i = k$.  

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In the next section we present our main filter SQP method. Each iteration requires the search for a new point that must satisfy a subset of specified conditions. We stress that the updated point \( x_{k+1} \) is not necessarily acceptable to \( F_k \). Moreover, the vector \( x_{k+1} \) being acceptable to \( F_k \) (possibly augmented by \( x_k \)) is a necessary, but not sufficient, condition for adding the ordered pair \((v_{k+1}, f_{k+1})\) to the filter \( F_k \). Details of how we update \( F_k \) are described in the next section.

2.8. The algorithm. Our method is formally stated as Algorithm 1. Every iteration begins by computing the set of trial steps \( \{s_k^a, s_k\} \) as described in sections 2.5 and 2.3. Once these trial steps are computed, we seek a step length \( \alpha_k \) such that for some \( \hat{s}_k \in \{s_k^a, s_k\} \) the step \( x_k + \alpha_k \hat{s}_k \) satisfies one of four possible sets of conditions. Which sets of conditions we seek to satisfy depends on whether the algorithm is in filter mode (roughly a traditional filter strategy) or penalty mode (our alternative to a traditional restoration phase). We now discuss these two modes in detail.

In filter mode, we perform a backtracking line search until we find a pair \((\alpha_k, \hat{s}_k)\) with \( \hat{s}_k \in \{s_k^a, s_k\} \) that forms a v-pair or an o-pair, or a pair \((\alpha_k, s_k)\) that forms a b-pair. We discuss these in turn.

A v-pair is defined as follows.

**Definition 2.14 (v-pair).** The pair \((\alpha, s)\) constitutes a v-pair if \( x_k + \alpha s \) is acceptable to \( F_k \) augmented by \( x_k \) and

\[
\Delta f^v(s_k; x_k) < \gamma_v \Delta v^v(s_k; x_k) \quad \text{for some} \quad \gamma_v \in (0, 1).
\]

A v-pair \((\alpha_k, \hat{s}_k)\) earns its name since the step \( x_k + \alpha_k \hat{s}_k \) is acceptable to the current filter augmented by \( x_k \), but the step \( s_k \) did not predict sufficient decrease in \( f \) as measured by (2.30); we say that \( k \) is a v-iterate since the focus of the iteration is on reducing the constraint violation \( v \). In this case, we choose to add the pair \((v_k, f_k)\) to the filter \( F_k \).

An o-pair is characterized as follows.

**Definition 2.15 (o-pair).** The pair \((\alpha, s)\) constitutes an o-pair if \( x_k + \alpha s \) is acceptable to \( F_k \),

\[
\begin{align*}
\Delta f^v(s_k; x_k) &\geq \gamma_v \Delta v^v(s_k; x_k) \quad \text{and} \\
f(x_k + \alpha s) \leq f(x_k) - \gamma_f \alpha \rho^f_k,
\end{align*}
\]

where \( \gamma_v \in (0, 1) \) is the same constant used to define a v-pair, \( \gamma_f \in (0, 1) \), and

\[
\rho^f_k := \min \left[ \Delta f^f(s_k; x_k), \Delta q^f(s_k^a; x_k, H_k) \right].
\]

An o-pair \((\alpha_k, \hat{s}_k)\) is so designated since \( x_k + \alpha_k \hat{s}_k \) is acceptable to the filter, \( s_k \) predicts decrease in the objective function as measured by (2.31a), and a sufficient decrease in the objective is realized as given by (2.31b); we say that \( k \) is an o-iterate since progress has been made on decreasing the objective function. In this case we do not add any new entries to the filter.

The definitions of v- and o-pairs are natural in light of the mechanism of the filter and are similar in spirit to conditions used by previous methods [13, 15, 21, 22, 24, 46]. As for these methods, these two sets of conditions are not sufficient for ensuring convergence since previously added filter entries may prevent (block) additional progress. In this situation, other filter algorithms typically decrease the trust-region radius or perform backtracking until a restoration phase is triggered. To prevent this undesirable situation we introduce the following definition of a b-pair.
Algorithm 1. Filter sequential quadratic programming algorithm.

1: Input an initial primal-dual pair \((x_0, y_0)\).
2: Choose \(\{\eta_\ell, \eta_\sigma, \eta_\phi, \sigma_{\text{inc}}, \beta, \gamma_\ell, \gamma_\sigma, \gamma_f, \gamma_\phi, \xi\} \subset (0, 1)\) and \(0 < \delta_{\min} \leq \delta_{\max} < \infty\).
3: Set \(k \leftarrow 0\), \(F_0 \leftarrow \emptyset\), \(P\)-mode \(\leftarrow\) false, and choose \(\sigma_0 > 0\) and \(\delta_0 \in [\delta_{\min}, \delta_{\max}]\).
4: loop
5: Compute \(s_k^\ell\) as a solution of (2.10), and then calculate \(\Delta \ell^\ell(s_k^\ell; x_k)\) from (2.6).
6: if \(\Delta \ell^\ell(s_k^\ell; x_k) = 0\) and \(v(x_k) > 0\), then
   \(\text{return}\) with the infeasible stationary point \(x_k\) for problem (1.1).
7: end if
8: Choose \(B_k > 0\) and compute \(s_k^\ell\) as the solution of (2.12) with multiplier \(y_k^\ell\).
9: if \(\Delta q^\ell(s_k^\ell; x_k, \sigma_k) = v(x_k) = 0\), then
   \(\text{return}\) with the KKT point \((x_k, y_k^\ell)\) for problem (1.1).
10: end if
11: Compute \(s_k = (1 - \tau_k) s_k^\ell + \tau_k s_k^\ell\) from (2.15) such that (2.16) is satisfied.
12: Compute the new weight \(\sigma_{k+1}\) from (2.19).
13: Choose \(\delta_k^a > 0\) and then compute \(s_k^a\) as an (approximate) solution of (2.25).
14: Compute \(s_k^{cf}\) from (2.28) and then calculate \(\Delta q^\ell(s_k^{cf}; x_k, H_k, \sigma_{k+1})\) from (2.9).
15: if \(P\)-mode then
   for \(j = 0, 1, 2, \ldots\) do
      16: Set \(\alpha_k \leftarrow \xi_j^a\).
      17: for \(\delta_k \in \{s_k^a, s_k\}\) do
         18: if \((\alpha_k, \delta_k)\) is a \(p\)-pair then
            19: Set \(F_{k+1} \leftarrow F_k\) and go to line 21.
            \(\triangleright p\)-iterate
         20: end if
      21: if \(x_k + \alpha_k \delta_k\) is acceptable to \(F_k\) then
         22: Set \(P\)-mode \(\leftarrow\) false.
         \(\triangleright\) p-iterate
      23: else
         24: Compute \(s_k^{cf}\) from (2.27) and then calculate \(\Delta q^f(s_k^{cf}; x_k, H_k)\) from (2.7b).
         for \(j = 0, 1, 2, \ldots\) do
            25: Set \(\alpha_k \leftarrow \xi_j^a\).
            26: for \(\delta_k \in \{s_k^a, s_k\}\) do
               27: if \((\alpha_k, \delta_k)\) is a \(v\)-pair then
                  28: Set \(F_{k+1} \leftarrow F_k \cup \{\langle v_k, f_k \rangle\}\) and go to line 34.
                  \(\triangleright v\)-iterate
               29: end if
               30: if \((\alpha_k, \delta_k)\) is an \(o\)-pair then
                  31: Set \(F_{k+1} \leftarrow F_k\) and go to line 34.
                  \(\triangleright o\)-iterate
               32: end if
               33: if \((\alpha_k, s_k)\) is a \(b\)-pair then
                  34: Set \(F_{k+1} \leftarrow F_k \cup \{\langle v_k, f_k \rangle\}\), \(P\)-mode \(\leftarrow\) true, and go to line 34.
                  \(\triangleright b\)-iterate
               35: end if
            36: end for
         37: end for
      38: end if
   39: end for
end if

Definition 2.16 (b-pair). The pair \((\alpha, s)\) constitutes a b-pair if

\[
(2.33) \quad v(x_k + \alpha s) < v(x_k)
\]

and

\[
(2.34) \quad \phi(x_k + \alpha s; \sigma_{k+1}) \leq \phi(x_k; \sigma_{k+1}) - \gamma_\phi \alpha \rho_k^\phi \quad \text{for some } \gamma_\phi \in (0, 1),
\]

where

\[
(2.35) \quad \rho_k^\phi := \min \left[ \Delta \ell^\ell(s_k; x_k, \sigma_{k+1}), \Delta q^\ell(s_k^{cf}; x_k, H_k, \sigma_{k+1}) \right].
\]
An iterate $x_k + \alpha_k s_k$ associated with a b-pair $(\alpha_k, s_k)$ decreases both the constraint violation and the penalty function; we say that $k$ is a b-iterate since the conditions that define a b-pair suggest that one or more filter entries are blocking a productive step. In this case, we choose to accept the trial point, add $(\nu_k, f_k)$ to the filter, and enter what we will refer to as a penalty mode. We view penalty mode as an alternative to a traditional restoration phase. Moreover, since steps are always tested for acceptability based on the filter criteria, i.e., o- and v-pairs, before checking for decrease in the constraint violation and penalty function as stipulated by b-pairs, we give clear preference to staying in filter mode.

In penalty mode, we calculate a new iterate by perform a backtracking line search until we find a pair $(\alpha_k, \hat{s}_k)$ for some $\hat{s}_k \in \{s^o_k, s_k\}$ that satisfies the following conditions that define a p-pair.

**Definition 2.17 (p-pair).** The pair $(\alpha, s)$ constitutes a p-pair if (2.34) is satisfied.

If $(\alpha_k, \hat{s}_k)$ is a p-pair, then $\phi(x_k + \alpha_k \hat{s}_k; \sigma_{k+1})$ is sufficiently smaller than $\phi(x_k; \sigma_{k+1})$; we say that $k$ is a p-iterate since the penalty function has been decreased. In addition, if $x_k + \alpha_k \hat{s}_k$ is acceptable to the current filter, we return to filter mode; otherwise, we remain in penalty mode.

Finally, after a new trial step is computed, we choose to increase the penalty parameter if

$$\Delta q^p(s_k; x_k, B_k, \sigma_{k+1}) < \eta_\phi \Delta q^p(s^o_k; x_k, B_k, \sigma_{k+1}) \quad \text{for some } \eta_\phi \in (0, 1),$$

since this indicates that $\tau_k$ is very small and the search direction $s_k$ does not adequately reflect the decrease predicted by $s^o_k$ in the penalty function.

For future reference we define the following index sets based on the different types of pairs:

$$S_v = \{ k : k \text{ is a v-iterate } \}, \quad S_o = \{ k : k \text{ is an o-iterate } \},$$

$$S_p = \{ k : k \text{ is a p-iterate } \}, \quad S_b = \{ k : k \text{ is an b-iterate } \}.$$

We complete this section by summarizing the computational complexity of each iteration of Algorithm 1, which requires the calculation of multiple directions. Specifically, each iteration requires the solution of a linear program to obtain the steering step (see (2.10)), a strictly convex quadratic program to get the predictor step (see (2.12)), a single matrix-vector multiplication to solve the one-dimensional optimization problem for the Cauchy-$f$ step (see (2.27)), a one-dimensional search along a piecewise linear path to obtain the Cauchy-$\phi$ step (see (2.28)), and an approximate solution to an equality-constrained quadratic problem for the an accelerator step (see (2.25)). Therefore, the predominant computational cost for each iteration is the calculation of the steering and predictor steps.

3. **Well-posedness.** In this section we verify that every step of the method is well-posed under the following assumption, which we do not explicitly state for each result.

**Assumption 3.1.** The functions $f$ and $c$ are both differentiable with Lipschitz continuous derivatives in the neighborhood of the point $x_k$.

We begin by observing that the steering problem (2.11) is convex and always feasible, and the objective function is bounded below by zero, i.e., it is well-defined. If $\nu(x_k) > 0$ and $\Delta f(s^o_k; x_k) = 0$, then $x_k$ is an infeasible stationary point and we exit in line 7 of Algorithm 1. Otherwise, $x_k$ is not an infeasible stationary point and
we proceed to compute a predictor step from problem (2.12), which we now argue is well-defined. This is obvious when \( \Delta \ell^v(s_k^0; x_k) \neq v(x_k) \) since then the strictly convex problem (2.12b) is always feasible. On the other hand, if \( \Delta \ell^v(s_k^0; x_k) = v(x_k) \), then it follows that \( \|c(x_k) + J(x_k)s_k^0\|_1 = 0 \), which implies that \( c_k + J_k s_k^0 \geq 0 \). Thus, \( s = s_k^0 \) is feasible for (2.12a), and the predictor problem is well-defined. Lemma 2.3 shows that \( \tau_k > 0 \) and Lemma 2.8 shows that the update to the weighting parameter is well-defined. The accelerator problem (2.25) does not cause difficulties since by construction it is feasible, has bounded solutions, and may be solved (approximately) as noted in section 2.5. It is also easy to see that both Cauchy step problems (2.27) and (2.28) are well-defined.

We now proceed to show that the line search terminates finitely. To this end, we first show that feasible iterates are never added to the filter.

**Lemma 3.1.** Algorithm 1 ensures that if \((v_k, f_k)\) is added to the filter, then \(v_k > 0\).

**Proof.** For a proof by contradiction, suppose that \(v(x_k) = 0\). It follows from \(v(x_k) = 0\) and the fact that \(\ell^v\) is a convex function that \(\Delta \ell^v(s_k^0; x_k) = 0\), and we may then use (2.12a), (2.15), (2.16), and the fact that \(x_k\) is not a KKT point for (1.1) (otherwise we would already have exited on line 10 of Algorithm 1) to show that

\[
\tau_k = 1, \quad s_k = s_k^0 \neq 0, \quad \text{and} \quad \Delta \ell^v(s_k; x_k) = \Delta \ell^v(s_k^0; x_k) = 0.
\]

It then follows from (3.1), (2.8), Lemma 2.7, \(v(x_k) = 0\), and the fact that \(B_k > 0\) that

\[
\Delta \ell^f(s_k; x_k) = \Delta \ell^f(s_k^0; x_k) = \Delta \ell^\phi(s_k; x_k, \sigma_{k+1}) = \Delta \ell^\phi(s_k; x_k, \sigma_{k+1}) \geq \frac{1}{2}s_k^T B_k s_k^0 > 0.
\]

Since \((v_k, f_k)\) was added to the filter, it follows from the construction of Algorithm 1 that either \((\alpha_k, s_k)\) is a \(v\)-iterate or \((\alpha_k, s_k)\) is a \(b\)-pair, which implies that at least one of \(v(x_k + \alpha_k s_k) < v(x_k)\) or \(\Delta \ell^f(s_k; x_k) < \gamma_v \Delta \ell^v(s_k; x_k)\) holds, among other requirements. However, since \(v(x_k + \alpha_k s_k) < v(x_k) = 0\) is not possible, we conclude that \(\Delta \ell^f(s_k; x_k) < \gamma_v \Delta \ell^v(s_k; x_k) = 0\), where we have also used (3.1); this contradicts (3.2) and proves the result. \(\square\)

The next two results show that our line search procedure terminates any time \(\mathcal{P}\text{-mode}\) has the value false at the beginning of the \(k\)th iteration. We first consider the case when \(x_k\) is feasible.

**Lemma 3.2.** If \(\mathcal{P}\text{-mode} = \text{false}\) at the beginning of the \(k\)th iteration, \(v(x_k) = 0\), and \(x_k\) is not a first-order solution to problem (1.1), then the pair \((\alpha, s_k)\) is an \(o\)-pair for all \(\alpha > 0\) sufficiently small. Moreover, \(k \in S_\alpha\).

**Proof.** As in the proof of Lemma 3.1, it follows that \(v(x_k) = \Delta \ell^v(s_k^0; x_k) = 0\). This may be combined with the fact that \(x_k\) is assumed to not be a first-order solution to (1.1), (2.12a), (2.15), (2.16), Lemma 2.7, the fact that \(B_k\) is positive definite, and the definition of \(\Delta \ell^\phi\) to conclude that

\[
\Delta \ell^f(s_k; x_k) = \Delta \ell^\phi(s_k; x_k, \sigma_{k+1}) > 0 = \gamma_v \Delta \ell^v(s_k; x_k).
\]

Next, \(v(x_k) = 0\) and (3.3) imply that \(c_k + \alpha J_k s_k \geq 0\) for all \(\alpha \in [0, 1]\). Combining this fact with Taylor’s theorem, Assumption 3.1, and (3.3) yields

\[
v(x_k + \alpha s_k) = \|c(x_k + \alpha s_k)\|_1 = \|c_k + \alpha J_k s_k + O(\alpha^2)\|_1 \leq O(\alpha^2) \quad \text{for} \quad \alpha \in [0, 1].
\]
Since Lemma 3.1 implies that \( v_i > 0 \) for all \((v_i, f_i) \in \mathcal{F}_k\), we may conclude from (3.4) that
\[
v(x_k + \alpha s_k) \leq \min_{(v_i, f_i) \in \mathcal{F}_k} \beta v_i \quad \text{for all } \alpha > 0 \text{ sufficiently small},
\]
where \( \beta \in (0, 1) \) is defined in (2.29), so that
\[
\tag{3.5} x_k + \alpha s_k \text{ is acceptable to the filter for all } \alpha > 0 \text{ sufficiently small}.
\]

Next, Taylor’s theorem, Assumption 3.1, the definition of \( \Delta \ell^f \), and (3.3) imply that
\[
f(x_k + \alpha s_k) = f_k + \alpha g_k^T s_k + O(\alpha^2) = f_k - \alpha \Delta \ell^f(s_k; x_k) + O(\alpha^2)
\]
\[
\tag{3.6} \leq f_k - \gamma_f \alpha \Delta \ell^f(s_k; x_k) \quad \text{for all } \alpha > 0 \text{ sufficiently small},
\]
where \( \gamma_f \in (0, 1) \) is defined in (2.31b). It follows from (3.3), (3.5), and (3.6) that \((\alpha, s_k)\) is an o-pair for all \( \alpha > 0 \) sufficiently small, which proves the first result of this lemma.

We just proved that the for loop on line 25 in Algorithm 1 always terminates. Moreover, it cannot terminate on line 28 since (3.3) holds. Also, it can never terminate as a result of the if on line 32 since \( v(x_k + \alpha s_k) < v(x_k) = 0 \) is impossible for all \( \alpha \). Therefore, the line search must terminate with an o-pair \((\alpha_k, \hat{s}_k)\), which implies that \( k \in \mathcal{S}_o \). \[\Box\]

We now consider the case when \( x_k \) is infeasible.

**Lemma 3.3.** If \( \mathcal{P} \)-mode = false at the beginning of iteration \( k \), \( v(x_k) > 0 \), and \( x_k \) is not an infeasible stationary point, then \((\alpha, s_k)\) is a b-pair for all \( \alpha > 0 \) sufficiently small.

**Proof.** It follows from the assumptions of this lemma and Lemma 2.10 that
\[
\tag{3.7} [D_{s_k} \phi](x_k; \sigma_{k+1}) \leq -\Delta \ell^\phi(s_k; x_k, \sigma_{k+1}) < 0
\]
so that the direction \( s_k \) is a strict descent direction for \( \phi \) at \( x_k \) with penalty parameter \( \sigma_{k+1} \). Using the definition of the directional derivative, (3.7), \( \gamma_\phi \in (0, 1) \) defined in (2.34), and (2.35) we conclude that
\[
\tag{3.8} \phi(x_k + \alpha s_k; \sigma_{k+1}) \leq \phi(x_k; \sigma_{k+1}) + \alpha \gamma_\phi [D_{s_k} \phi](x_k; \sigma_{k+1})
\]
\[
\leq \phi(x_k; \sigma_{k+1}) - \alpha \gamma_\phi \Delta \ell^\phi(s_k; x_k, \sigma_{k+1})
\]
\[
\leq \phi(x_k; \sigma_{k+1}) - \alpha \gamma_\phi \rho_k^\phi \quad \text{for all } \alpha > 0 \text{ sufficiently small}.
\]

Since \( v(x_k) \neq 0 \) and \( x_k \) is not an infeasible stationary point, we know that \( \Delta \ell^v(s_k^*; x_k) > 0 \). Lemma 2.6 then implies that
\[
[D_{s_k} v](x_k) \leq -\Delta \ell^v(s_k; x_k) \leq -\eta_v \Delta \ell^v(s_k^*; x_k) < 0
\]
so that \( s_k \) is a descent direction for \( v \) at \( x_k \). An argument similar to the one that lead to (3.8) yields
\[
\tag{3.9} v(x_k + \alpha s_k) < v(x_k) \quad \text{for all } \alpha > 0 \text{ sufficiently small}.
\]

It follows from (3.8) and (3.9) that \((\alpha, s_k)\) is a b-pair for all \( \alpha > 0 \) sufficiently small. \[\Box\]

The next lemma considers the case when \( \mathcal{P} \)-mode is true at the beginning of the \( k \)th iteration and shows that successful trial iterates may be obtained through backtracking as performed in Algorithm 1.
**Lemma 3.4.** If $\mathcal{P}$-mode = true at the beginning of the $k$th iteration and $x_k$ is neither an infeasible stationary point nor a first-order solution to problem (1.1), then $(\alpha, s_k)$ is a $p$-pair for all $\alpha > 0$ sufficiently small.

**Proof.** The proof follows exactly as in the first part of Lemma 3.3.  

We now combine these results to prove that Algorithm 1 is well-posed.

**Theorem 3.5.** Algorithm 1 is well-posed.

**Proof.** As described in the first paragraph of section 3, every subproblem and step computation is well defined, and Lemma 2.8 ensures that the update to the weighting parameter is well defined.

All that remains is to prove that the line search terminates. First, if $\mathcal{P}$-mode has the value false at the beginning of iteration $x_k$ and $v(x_k) = 0$, then Lemma 3.2 guarantees finite termination and that $k \in S_o$. Second, if $\mathcal{P}$-mode has the value false and $v(x_k) > 0$, then Lemma 3.3 ensures that the backtracking line search will terminate finitely. Finally, suppose that $\mathcal{P}$-mode has the value true at the beginning of iteration $k$. It then follows from Lemma 3.4 that the backtracking terminates finitely.  

4. Global convergence. In this section we prove that limit points of the iterates generated by Algorithm 1 have desirable properties. To this end, we use the following common assumptions.

**Assumption 4.1.** The iterates $\{x_k\}$ lie in an open, bounded, convex set $X$.

**Assumption 4.2.** The problem functions $f(x)$ and $c(x)$ are twice continuously differentiable on $X$.

**Assumption 4.3.** The matrices $B_k$ are uniformly positive definite and bounded, i.e., there exists values $0 < \lambda_{\min} < \lambda_{\max} < \infty$ such that $\lambda_{\min} \|s\|^2 \leq s^TB_k s \leq \lambda_{\max} \|s\|^2$ for all $s \in \mathbb{R}^n$ and all $B_k$.

**Assumption 4.4.** The matrices $H_k$ are uniformly bounded, i.e., $\|H_k\|_2 \leq \mu_{\max}$ for some $\mu_{\max} \geq 1$.

For clarity and motivational purposes, we immediately state our main convergence theorem, which makes use of the Mangasarian–Fromovitz constraint qualification (MFCQ) [35].

**Theorem 4.1.** If Assumptions 4.1–4.4 hold, then one of the following must occur:

(i) Algorithm 1 terminates finitely with either a first-order KKT point or an infeasible stationary point in lines 10 or 7, respectively, for problem (1.1).

(ii) Algorithm 1 generates infinitely many iterations $\{x_k\}$, $\sigma_k = \sigma < \infty$ for all $k$ sufficiently large, and there exists a limit point $x_*$ of $\{x_k\}$ that is either a first-order KKT point or an infeasible stationary point for problem (1.1).

(iii) Algorithm 1 generates infinitely many iterations $\{x_k\}$, $\lim_{k \to \infty} \sigma_k = \infty$, and there exists a limit point $x_*$ of $\{x_k\}$ that is either an infeasible stationary point or a feasible point at which the MFCQ fails.

**Proof.** The result follows from the following analysis that considers the various cases that can occur. In particular, it follows from Theorems 4.11, 4.14, 4.18, and 4.21 and the construction of Algorithm 1.

We now present a sequence of lemmas that will be useful in the convergence analysis. The first result is adapted from [9, Theorem 3.6] and provides a bound on the trial step $s_k$.

**Lemma 4.2.** If Assumptions 4.1–4.3 hold and $x_k$ and $s_k$ are generated by Algorithm 1, then

$$
\|s_k\|_2 \leq \max \left\{1, \frac{2}{\lambda_{\min}} \left[\|g_k\|_2 + \sigma_k v(x_k)\right], \sqrt{\mu_{\max}}\right\}.
$$

4.21 and the construction of Algorithm 1.
Furthermore, if \{\sigma_k\} is bounded, then there exists a constant \(M_\alpha > 0\) such that \(\|s_k\|_2 \leq M_\alpha\) for all \(k\).

Proof. First, we claim that the predictor step \(s_k^p\) must satisfy

\[
\|s_k^p\|_2 \leq \max \left\{ 1, \frac{2}{\lambda_{\min}} \left[ \|g_k\|_2 + \sigma_k v(x_k) \right] \right\},
\]

which can be seen as follows. Suppose that (4.2) is not satisfied so that

\[
\|s_k^p\|_2 > 1 \quad \text{and} \quad \frac{1}{2} \lambda_{\min} \|s_k^p\|_2 > \|g_k\|_2 + \sigma_k v(x_k).
\]

It then follows from the definitions of \(\Delta q^p\) and \(\ell^v\), the Cauchy–Schwarz inequality, Assumption 4.3, and (4.3) that

\[
\Delta q^p(s_k^p; x_k, B_k, \sigma_k) = -g_k^T s_k^p - \frac{1}{2} s_k^p T B_k s_k^p + \sigma_k (\ell^v(0; x_k) - \ell^v(s_k^p; x_k))
\]

\[
\leq \|g_k\|_2 \|s_k^p\|_2 - \frac{1}{2} \lambda_{\min} \|s_k^p\|_2^2 + \sigma_k v(x_k)
\]

\[
\leq \|g_k\|_2 \|s_k^p\|_2 - \frac{1}{2} \lambda_{\min} \|s_k^p\|_2^2 + \|s_k^p\|_2 \sigma_k v(x_k)
\]

\[
= \|s_k^p\|_2 \left( \|g_k\|_2 - \frac{1}{2} \lambda_{\min} \|s_k^p\|_2 + \sigma_k v(x_k) \right) < 0,
\]

which contradicts the fact that \(s_k^p\) is the unique global minimizer to the strictly convex predictor problem. Thus, (4.2) must hold and when combined with (2.15), the use of the triangle inequality, and the use of the trust-region radius \(\delta_k \in [\delta_{\min}, \delta_{\max}]\) in the steering problem implies that

\[
\|s_k\|_2 \leq \max \left\{ 1, \frac{2}{\lambda_{\min}} \left[ \|g_k\|_2 + \sigma_k v(x_k) \right] \sqrt{\delta_{\max}} \right\},
\]

which proves (4.1). Since \(g_k\) and \(v(x_k)\) are uniformly bounded as a result of Assumptions 4.1 and 4.2, it is clear that if \{\sigma_k\} is bounded, then there exists \(M_\alpha < \infty\) such that \(\|s_k\|_2 \leq M_\alpha\) for all \(k\).

The following result provides a relationship between the predicted change in the linear model and the change achieved in the line search process for both the objective function and the constraint violation.

Lemma 4.3 (equivalent to [45, Lemma 3]). Suppose that Assumptions 4.1 and 4.2 hold. Then, there exist constants \(\{C_f, C_v\} > 0\) such that for all \(k\) and \(\alpha \in (0, 1]\), we have

\[
f(x_k + \alpha s) \leq f(x_k) - \alpha \Delta \ell^f(s; x_k) + \alpha^2 C_f \|s\|_2^2
\]

and

\[
v(x_k + \alpha s) \leq v(x_k) - \alpha \Delta \ell^v(s; x_k) + \alpha^2 C_v \|s\|_2^2.
\]

Proof. Inequality (4.5) is a direct result of Taylor’s theorem and Assumption 4.2.

For (4.6), it follows from the integral mean-value theorem, Assumptions 4.1 and 4.2, and the implied Lipschitz continuity of \(J(x)\), the triangle inequality, and the convexity of \(\ell^v\), that for some constant Lipschitz constant \(C_v\),
\[ v(x_k + \alpha s) \]
\[ = \| c(x_k + \alpha s) \|_1 = \left\| c(x_k) + \alpha J_k s + \alpha \int_0^1 [J(x_k + \theta \alpha s) - J(x_k)] s d\theta \right\|_1 \]
\[ \leq \| c(x_k) + \alpha J_k s \|_1 + \alpha^2 \sqrt{n} C \| s \|_2^2 \]
\[ \leq (1 - \alpha) \| c(x_k) \|_1 + \alpha \| c(x_k) + J_k s \|_1 + \alpha^2 \sqrt{n} C \| s \|_2^2 \]
\[ = v(x_k) - \alpha \Delta f^\ell(s_k; x_k) + \alpha^2 \sqrt{n} C \| s \|_2^2 \text{ for all } \alpha \in (0, 1). \]

This proves (4.6) by defining \( C_v := \sqrt{n} C. \quad \square \)

The next two lemmas provide a relationship between the predicted linear decrease in the objective function and the quantity \( \rho_k^\ell \) defined by (2.32).

**Lemma 4.4.** If Assumption 4.4 holds and \( \Delta f^\ell(s_k; x_k) \geq 0 \), then

\[
\Delta q^\ell(s_k^\ell; x_k, H_k) \geq \frac{1}{2} \Delta f^\ell(s_k; x_k) \min \left\{ \frac{\Delta f^\ell(s_k; x_k)}{\mu_{\max} \| s_k \|_2^2}, 1 \right\}.
\]

**Proof.** If \( \Delta f^\ell(s_k; x_k) = 0 \), then the result follows immediately from the definition of \( s_k^\ell \) in (2.27).

Now, suppose that \( \Delta f^\ell(s_k; x_k) > 0 \). It follows from (2.27) and the definition of \( \Delta q^\ell \) that

\[
\Delta q^\ell(s_k^\ell; x_k, H_k) \geq \Delta q^\ell(\alpha s_k; x_k, H_k) = -\alpha q^\ell T s_k - \frac{1}{2} \alpha^2 s_k^T H_k s_k \text{ for all } 0 \leq \alpha \leq 1.
\]

The right-hand side of the previous equation may be written as

\[
q(\alpha) = a\alpha^2 + b\alpha, \text{ where } a = -\frac{1}{2} s_k^T H_k s_k \text{ and } b = \Delta f^\ell(s_k; x_k) = -q^\ell T s_k > 0.
\]

We wish to maximize \( q \) on the interval \([0, 1]\) so we differentiate \( q(\alpha) \) with respect to \( \alpha \) and set the result to zero to obtain a stationary point at \(-\frac{b}{2a}\). Now, consider three cases.

**Case 1** \((a < 0 \text{ and } -\frac{b}{2a} \leq 1)\). The maximum of \( q(\alpha) \) on the interval \([0, 1]\) is achieved at \( \alpha = -\frac{b}{2a} \). Note that \( \alpha > 0 \), since \( b = \Delta f^\ell(s_k; x_k) > 0 \) by assumption. Then, we have

\[
q\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 - b\frac{b}{2a} = -\frac{b^2}{4a^2}.
\]

It follows from the definition of \( a \) and \( b \), the Cauchy–Schwarz inequality, and Assumption 4.4 that

\[
q\left(-\frac{b}{2a}\right) = \frac{\Delta f^\ell(s_k; x_k)^2}{2s_k^T H_k s_k} \geq \frac{\Delta f^\ell(s_k; x_k)^2}{2\|H_k\|_2 \|s_k\|_2} \geq \frac{\Delta f^\ell(s_k; x_k)^2}{2\mu_{\max} \|s_k\|_2^2}.
\]

**Case 2** \((a < 0 \text{ and } -\frac{b}{2a} > 1)\). The maximum of \( q(\alpha) \) on the interval \([0, 1]\) is achieved at \( \alpha = 1 \), where

\[
q(1) = a + b > -\frac{1}{2} b + b = \frac{1}{2} b = \frac{1}{2} \Delta f^\ell(s_k; x_k).
\]

**Case 3** \((a \geq 0)\). The maximum of \( q(\alpha) \) on the interval \([0, 1]\) is achieved at \( \alpha = 1 \) so that

\[
q(1) = a + b > b > \frac{1}{2} b = \frac{1}{2} \Delta f^\ell(s_k; x_k).
\]
Finally, combining all three cases and defining $\alpha' = \arg \max_{\alpha \in [0,1]} q(\alpha)$, it follows that

$$
\Delta q^f(s_k^c; x_k, H_k) = q(\alpha') \geq \min \left\{ \frac{\Delta^f(s_k; x_k)^2}{2\mu_{\text{max}}}, \frac{1}{2} \Delta^f(s_k; x_k) \right\}
$$

$$
= \frac{1}{2} \Delta^f(s_k; x_k) \min \left\{ \frac{\Delta^f(s_k; x_k)}{\mu_{\text{max}}}, 1 \right\}
$$

as desired. □

**Lemma 4.5.** Suppose that Assumptions 4.1–4.4 are satisfied and that $\{\sigma_k\}$ is bounded. Then, there exists a constant $C_\rho > 0$ such that whenever $\Delta^f(s_k; x_k) \geq 0$, it follows that

$$
\rho_k^f \geq \min \left\{ C_\rho \Delta^f(s_k; x_k)^2, \frac{1}{2} \Delta^f(s_k; x_k) \right\}.
$$

**Proof.** It follows from (2.32), Lemma 4.4, Lemma 4.2, and the assumption $\Delta^f(s_k; x_k) \geq 0$ that

$$
\rho_k^f = \min \left\{ \Delta^f(s_k; x_k), \Delta q^f(s_k^c; x_k) \right\}
$$

$$
\geq \min \left\{ \Delta^f(s_k; x_k), \min \left\{ \frac{\Delta^f(s_k; x_k)^2}{2\mu_{\text{max}}}, \frac{1}{2} \Delta^f(s_k; x_k) \right\} \right\}
$$

$$
\geq \min \left\{ \frac{\Delta^f(s_k; x_k)^2}{2\mu_{\text{max}}} \frac{1}{\mu_{\text{max}}} \Delta^f(s_k; x_k) \right\} \geq \min \left\{ \frac{\Delta^f(s_k; x_k)^2}{2\mu_{\text{max}} M_\sigma^2}, \frac{1}{2} \Delta^f(s_k; x_k) \right\},
$$

where $\{M_\sigma, \mu_{\text{max}}\} \subset (0, \infty)$ are defined in (4.1) and Assumption 4.4, respectively. The result now follows by defining $C_\rho := 1/(2\mu_{\text{max}} M_\sigma^2)$. □

The next two results provide a relationship between the predicted linear change in the penalty function and the quantity $\rho_k^f$ defined by (2.35).

**Lemma 4.6.** If Assumption 4.4 holds and $x_k$ is not an infeasible stationary point, then

$$
\Delta q^\phi(s_k^c; x_k, H_k, \sigma_k+1) \geq \frac{1}{2} \Delta^\phi(s_k; x_k, \sigma_k+1) \min \left\{ \frac{\Delta^\phi(s_k; x_k, \sigma_k+1)}{\mu_{\text{max}}}, 1 \right\}.
$$

**Proof.** Since $x_k$ is not an infeasible stationary point by assumption, it follows from Lemma 2.8 that $\Delta^\phi(s_k; x_k, \sigma_k+1) \geq 0$. If $\Delta^\phi(s_k; x_k, \sigma_k+1) = 0$, then the result follows immediately. Therefore, for the remainder of the proof we assume that $\Delta^\phi(s_k; x_k, \sigma_k+1) > 0$.

It follows from (2.28), the convexity of $\ell^\phi(\cdot)$, and simple algebra that

$$
\Delta q^\phi(s_k^c; x_k, H_k, \sigma_k+1) \geq \Delta q^\phi(s_k^c; x_k, H_k, \sigma_k+1)
$$

$$
= -\alpha q^T_k s_k - \frac{1}{2} \alpha^2 s_k^T H_k s_k + \sigma_k+1 \left( \|c_k\|_1 - \|c_k + \alpha J_k s_k\|_1 \right)
$$

$$
\geq -\alpha q^T_k s_k - \frac{1}{2} \alpha^2 s_k^T H_k s_k + \sigma_k+1 \left( \|c_k\|_1 - \alpha \|c_k + J_k s_k\|_1 \right) - (1 - \alpha) \|c_k\|_1
$$

$$
= -\alpha q^T_k s_k - \frac{1}{2} \alpha^2 s_k^T H_k s_k + \alpha \sigma_k+1 \left( \|c_k\|_1 - \|c_k + J_k s_k\|_1 \right)
$$

$$
= \alpha \Delta^\phi(s_k; x_k, \sigma_k+1) - \frac{1}{2} \alpha^2 s_k^T H_k s_k \text{ for all } \alpha \in [0, 1].
$$
The right-hand side of the equation is a quadratic function of $\alpha$:

$$q(\alpha) = a\alpha^2 + b\alpha,$$

where $a = -\frac{1}{2} s_k^T H_k s_k$ and $b = \Delta \ell(\sigma_k; x_k, \sigma_{k+1}) > 0$.

Analysis similar to that used in the proof of Lemma 4.4 yields

$$\Delta q^\phi(s_k^\phi; x_k, H_k, \sigma_{k+1}) \geq \min \left\{ \frac{\Delta \ell(\sigma_k; x_k, \sigma_{k+1})^2}{2\mu_{\max} ||s_k||^2_2}, \frac{1}{2} \Delta \ell(\sigma_k; x_k, \sigma_{k+1}) \right\},$$

where $\mu_{\max}$ is from Assumption 4.4, as desired.

**Lemma 4.7.** Suppose that Assumptions 4.1–4.4 are satisfied, that Algorithm 1 never encounters an infeasible stationary point, and that $\{\sigma_k\}$ is bounded. Then, there exists a constant $C_\rho \in (0, \infty)$ such that

$$\rho^\phi_k \geq \min \left\{ C_\rho \Delta \ell(\sigma_k; x_k, \sigma_{k+1}), \frac{1}{2} \Delta \ell(\sigma_k; x_k, \sigma_{k+1}) \right\} \quad \text{for all } k \geq 0.$$

**Proof.** The proof follows exactly as in Lemma 4.5.

### 4.1. Convergence analysis under bounded weighting parameter.

In this section we study Algorithm 1 under the assumption that the weighting parameter stays bounded. It follows from this assumption and Lemma 4.2 that there exists some $k'$ and $\bar{\sigma} < \infty$ such that

$$||s_k||_2 \leq M_s < \infty \quad \text{and} \quad \sigma_k = \bar{\sigma} < \infty \quad \text{for all } k \geq k'.$$

Part (iii) of Theorem 4.1 implies that this scenario is guaranteed to occur, for example, when all limit points are neither infeasible stationary points nor feasible points at which the MFCQ fails.

We begin by showing that the line search step length is bounded away from zero in certain situations.

**Lemma 4.8.** If Assumptions 4.1–4.3 and (4.11) hold and $\epsilon > 0$, then the following hold:

(i) There exists a constant $\alpha_\rho > 0$ such that $\alpha_k \geq \alpha_\rho > 0$ for all $k \in K_\rho$, where

$$K_\rho = \{ k \in S_\rho : k \geq k' \quad \text{and} \quad \Delta \ell^\phi(s_k; x_k, \bar{\sigma}) \geq \epsilon \}.$$

(ii) There exists a constant $\alpha_\ell > 0$ such that $\alpha_k \geq \alpha_\ell > 0$ for all $k \in K_\ell$, where

$$K_\ell = \{ k \in S_o \cup S_o \cup S_i : k \geq k' \quad \text{and} \quad \Delta \ell^\ell(s_k; x_k) \geq \epsilon \}.$$

(iii) There exists a constant $\alpha_\ell > 0$ such that $(\alpha, s) = (\alpha, s_k)$ satisfies (2.31b) for all $0 < \alpha \leq \alpha_\ell$ and all $k \in K_\ell$, where

$$K_\ell = \{ k \geq k' : \Delta \ell^\ell(s_k; x_k) \geq \epsilon \}.$$

**Proof.** From [8, Lemma 3.4], there exists some positive constant $C_\phi$ such that

$$|\phi(x_k + \alpha s_k; \bar{\sigma}) - \ell^\phi(\alpha s_k; x_k, \bar{\sigma})| \leq C_\phi ||\alpha s_k||^2_2 \quad \text{for all } k \geq k' \quad \text{and} \quad \alpha \in [0, 1].$$

We first prove part (i). Suppose that $\alpha$ satisfies

$$0 \leq \alpha \leq \frac{(1 - \gamma_\phi)\epsilon}{C_\phi M^2_f},$$

**Lemma 4.9.** Suppose that $\alpha$ satisfies

$$0 \leq \alpha \leq \frac{(1 - \gamma_\phi)\epsilon}{C_\phi M^2_f},$$

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where $\gamma_\phi \in (0, 1)$ is set in Algorithm 1 and $M_s$ is defined in (4.11). We then use 
$\phi(x_k; \bar{\sigma}) = \ell_\phi(0; x_k, \bar{\sigma})$, the convexity of $\ell_\phi(\cdot; x_k, \bar{\sigma})$, (4.12), $\Delta \ell_\phi(s_k; x_k, \bar{\sigma}) \geq \epsilon$ for $k \in K_F$, (4.11), (4.13), and (2.35) to conclude that

$$
\phi(x_k; \bar{\sigma}) - \phi(x_k + \alpha s_k; \bar{\sigma}) \\
= \left[\ell_\phi(0; x_k, \bar{\sigma}) - \ell_\phi(\alpha s_k; x_k, \bar{\sigma})\right] - \left[\phi(x_k + \alpha s_k; \bar{\sigma}) - \ell_\phi(\alpha s_k; x_k, \bar{\sigma})\right] \\
\geq \left[\ell_\phi(0; x_k, \bar{\sigma}) - \alpha \ell_\phi(s_k; x_k, \bar{\sigma}) - (1 - \alpha)\ell_\phi(0; x_k, \bar{\sigma})\right] - C_\phi \alpha^2 \|s_k\|^2 \\
= \alpha \left[\ell_\phi(0; x_k, \bar{\sigma}) - \ell_\phi(s_k; x_k, \bar{\sigma})\right] - C_\phi \alpha^2 \|s_k\|^2 \\
= \gamma_\phi \alpha \Delta \ell_\phi(s_k; x_k, \bar{\sigma}) + (1 - \gamma_\phi) \alpha \Delta \ell_\phi(0; x_k, \bar{\sigma}) - C_\phi \alpha^2 \|s_k\|^2 \\
\geq \gamma_\phi \alpha \Delta \ell_\phi(s_k; x_k, \bar{\sigma}) + (1 - \gamma_\phi) \alpha \epsilon - C_\phi \alpha^2 \|s_k\|^2 \\
\geq \gamma_\phi \alpha \Delta \ell_\phi(s_k; x_k, \bar{\sigma}) \geq \gamma_\phi \alpha \rho_k^\phi \text{ for all } k \in K_F,$$

which with (2.34) implies that $(\alpha, s_k)$ is a $p$-pair. Thus, Algorithm 1 must select an 
$\alpha_k$ that satisfies

$$
(4.14) \quad \alpha_k \geq \min \left\{ \frac{\xi(1 - \gamma_\phi) \epsilon}{C_\phi M_s^2}, 1 \right\} =: \alpha_v,
$$

where $\xi \in (0, 1)$ is the backtracking parameter in Algorithm 1, which completes the 
proof of part (i).

We now prove part (ii). It follows from (2.20) that

$$
(4.15) \quad \Delta \ell_\phi(s_k; x_k, \bar{\sigma}) \geq \sigma_\pi \Delta \ell_\phi(s_k^\pi; x_k) \geq \bar{\sigma} \eta_\epsilon \epsilon \text{ for } k \in K_F.
$$

If $\alpha$ satisfies

$$
(4.16) \quad \alpha \leq \min \left[ \frac{(1 - \gamma_\phi) \bar{\sigma} \eta_\epsilon \epsilon}{C_\phi M_s^2}, \frac{\eta_\epsilon \epsilon}{C_v M_s^2} \right],
$$

where $C_v$ is defined in (4.6) and $\eta_\epsilon$ is defined in (2.16), then we may use (4.15) and

$$
(4.17) \quad v(x_k + \alpha s_k) - v(x_k) \\
\leq -\alpha \Delta \ell_v(s_k; x_k) + \alpha^2 C_v \|s_k\|^2 \leq -\alpha \eta_\epsilon \Delta \ell_\phi(s_k^\pi; x_k) + \alpha \frac{\eta_\epsilon \epsilon}{C_v M_s} C_v \|s_k\|^2 \\
\leq -\alpha \eta_\epsilon \epsilon + \alpha \eta_\epsilon \epsilon = 0 \text{ for all } k \in K_F,
$$

where the strict inequality holds since $s_k \neq 0$ as a result of (4.15). Combining (4.17) with (2.34) implies that $(\alpha, s_k)$ is a $b$-pair. Thus, we conclude from the structure of 
Algorithm 1 that

$$
(4.18) \quad \alpha_k \geq \min \left\{ \frac{\xi(1 - \gamma_\phi) \bar{\sigma} \eta_\epsilon \epsilon}{C_\phi M_s^2}, \frac{\xi \eta_\epsilon \epsilon}{C_v M_s^2}, 1 \right\} := \alpha_v > 0 \text{ for all } k \in K_F,
$$

where $\xi \in (0, 1)$ is the backtracking parameter used in Algorithm 1.

Part (iii) is a standard result used in continuous unconstrained optimization that 
follows since $\Delta \ell_\phi(s_k; x_k) \geq \epsilon$ is equivalent to $g(x_k)^T s_k \leq -\epsilon < 0$ and $s_k$ is uniformly 
bounded by (4.11).
The next lemma justifies the three cases that we consider when analyzing Algorithm 1.

**Lemma 4.9.** If Algorithm 1 does not terminate finitely, then one of the following scenarios occurs:

- **Case 1.** \( k \in S_p \) for all \( k \) sufficiently large,
- **Case 2.** \( k \in S_o \) for all \( k \) sufficiently large, or
- **Case 3.** \( |S_p \cup S_o| = \infty. \)

**Proof.** We proceed by contradiction and assume that none of the cases occurs. In particular, since Case 3 does not hold it follows that \( k \in S_p \cup S_o \) for all \( k \) sufficiently large. Combining this with the fact that Cases 1 and 2 do not hold implies that the iterates must oscillate between \( p \)- and \( o \)-iterates. However, this is not possible since there is no mechanism in Algorithm 1 that allows for iterate \( k+1 \) to be a \( p \)-iterate if iterate \( k \) is an \( o \)-iterate. \( \square \)

We now analyze Algorithm 1 for each of the three possible scenarios stated in the previous result.

**Case 1:** \( k \in S_p \) for all \( k \) sufficiently large. In this case, there exists \( k'' \) such that

\[
(4.19) \quad k \in S_p \quad \text{for all} \quad k \geq k'' \geq k',
\]

where \( k' \) is defined in \((4.11)\). We first show that our measure of feasibility converges to zero.

**Lemma 4.10.** If Assumptions 4.1–4.4, \((4.11)\), and \((4.19)\) hold, then \( \lim_{k \to \infty} \Delta^\ell(v)(s_k^x; x_k) = 0 \).

**Proof.** For a proof by contradiction, suppose that there exists an infinite subsequence \( S'' := \{k \geq k'' : \Delta^\ell(v)(s_k^x; x_k) \geq \epsilon''\} \) for some constant \( \epsilon'' > 0 \). It follows from \((4.19)\), \((2.20)\), \((4.11)\), and the definition of \( S'' \) that

\[
(4.20) \quad \Delta^\ell(v)(s_k^x; x_k, \bar{\sigma}) \geq \bar{\sigma} \eta_{\alpha} \Delta^\ell(v)(s_k^x; x_k) \geq \bar{\sigma} \eta_{\alpha} \epsilon'' =: \epsilon > 0 \quad \text{for all} \quad k \in S'',
\]

which implies with \((4.19)\) that

\[
S'' \subseteq K_P := \{k \in S_p : k \geq k' \text{ and } \Delta^\ell(v)(s_k^x; x_k, \bar{\sigma}) \geq \epsilon > 0\}.
\]

Combining \( K_P \) with Lemma 4.8 implies the existence of a positive \( \alpha_P \) such that \( \alpha_k \geq \alpha_P > 0 \) for all \( k \in S'' \), which used with the definitions of \( S'' \) and \( S_p \), \((4.19)\), Lemma 4.7, and \((4.20)\) yields

\[
(4.21) \quad \phi(x_k; \bar{\sigma}) - \phi(x_k + \alpha_k \bar{s}_k; \bar{\sigma}) \geq \gamma_{\phi} \alpha_k \bar{\rho}^\phi_k \\
\geq \gamma_{\phi} \alpha_k \min \left[C_p \Delta^\ell(v)(s_k^x; x_k, \bar{\sigma}), \frac{1}{2} \Delta^\ell(v)(s_k^x; x_k, \bar{\sigma})\right] \\
\geq \gamma_{\phi} \alpha_k \min \left[C_p \epsilon^2, \frac{1}{2} \epsilon\right] > 0 \quad \text{for all} \quad k \in S''.
\]

Now, for \( k'' \leq k \in S_p \setminus S'' \), it follows from \((2.34)\), \((2.20)\), and \((2.28)\) that

\[
(4.22) \quad \phi(x_k; \bar{\sigma}) - \phi(x_k + \alpha_k \bar{s}_k; \bar{\sigma}) \geq \gamma_{\phi} \alpha_k \min \left[\Delta^\ell(v)(s_k^x; x_k, \bar{\sigma}), \Delta q^\ell(v)(s_k^x; x_k, H_k, \bar{\sigma})\right] \geq 0.
\]

It is now easy to see from \((4.21)\), \((4.22)\), and \((4.19)\) that \( \lim_{k \to \infty} \phi(x_k; \bar{\sigma}) = -\infty \), which contradicts Assumptions 4.1 and 4.2. Thus, we conclude that \( \lim_{k \to \infty} \Delta^\ell(v)(s_k^x; x_k) = 0 \). \( \square \)
We now show that all limit points are infeasible stationary points for problem (1.1).

**Theorem 4.11.** Suppose that Assumptions 4.1–4.4, (4.11), and (4.19) hold. If \( x_\ast \) is any limit point of the sequence \( \{x_k\} \) generated by Algorithm 1, then \( x_\ast \) is an infeasible stationary point for problem (1.1).

**Proof.** Let \( v_{\min} := \min\{v_j : (v_j, f_j) \in F_k\} \equiv \min\{v_j : (v_j, f_j) \in F_k \text{ and } k \geq k''\} \), where the second equality holds since by assumption \( k \in S_p \) for all \( k \geq k'' \) and the filter is never expanded when \( k \in S_p \). It follows from Lemma 3.1 that \( v_{\min} > 0 \).

But then if there was a feasible limit point \( x_\ast \), there must be iterates \( x_k, k > k'' \) that are arbitrarily close to feasibility and thus ultimately one such that \( x_k \) is acceptable to \( F_k \). Thus line 21 of Algorithm 1 implies that there will be an iterate \( k > k'' \) for which \( k \notin S_p \), which contradicts (4.19). Thus, all limit points are infeasible. It follows from this fact, Lemma 4.10, and Lemma 2.1 that all limit points are infeasible stationary points.

Importantly, the previous result shows that our algorithm remains in penalty mode for all \( k \) sufficiently large only when all limit points are infeasible stationary points.

**Case 2:** \( k \in S_o \) for all \( k \) sufficiently large. In this case, there exists \( k'' \) such that

\[
(4.23) \quad k \in S_o \quad \text{for all } k \geq k'' \geq k',
\]

where \( k' \) is defined in (4.11). We begin by showing that our feasibility measure converges to zero.

**Lemma 4.12.** If Assumptions 4.1–4.4, (4.11), and (4.23) hold, then \( \lim_{k \to \infty} \Delta \ell^v(s_k^t; x_k) = 0 \).

**Proof.** For a contradiction, suppose that there exists \( \epsilon'' > 0 \) and an infinite subsequence

\[
S'' := \{k \geq k'' : \Delta \ell^v(s_k^t; x_k) \geq \epsilon''\} \subseteq S_o,
\]

where we have used \( k'' \) defined in (4.23). It then follows from the definition of \( S_o \), the o-pair \( (\alpha_k, \tilde{s}_k) \) selected in Algorithm 1, (2.31b), (4.11), Lemma 4.5, (2.31a), (2.16), and part (ii) of Lemma 4.8 that

\[
f(x_k) - f(x_k + \alpha_k \tilde{s}_k) - \gamma_f \alpha_k \rho_k^f \geq \gamma_f \alpha_k \min \left\{ C_\rho \Delta \ell^f(s_k; x_k)^2, \frac{1}{2} \Delta \ell^f(s_k; x_k) \right\} \\
\geq \gamma_f \alpha_k \min \left\{ C_\rho \gamma_v \Delta \ell^v(s_k; x_k)^2, \frac{1}{2} \gamma_v \Delta \ell^v(s_k; x_k) \right\} \\
\geq \gamma_f \alpha_k \min \left\{ C_\rho \gamma_v \eta_v \Delta \ell^v(s_k^t; x_k)^2, \frac{1}{2} \gamma_v \eta_v \Delta \ell^v(s_k^t; x_k) \right\} \\
\geq \gamma_f \alpha_k \min \left\{ C_\rho \gamma_v \eta_v \epsilon''^2, \frac{1}{2} \gamma_v \eta_v \epsilon'' \right\} \quad \text{for all } k \in S''
\]

for some \( \alpha_F > 0 \). Similarly, for \( k'' \leq k \in S_o \setminus S'' \), it follows from (2.31), (2.16), and (2.27) that

\[
f(x_k) - f(x_k + \alpha_k \tilde{s}_k) - \gamma_f \alpha_k \rho_k^f \geq \gamma_f \alpha_k \min \left\{ \gamma_v \Delta \ell^v(s_k; x_k), \Delta q^f(s_k^t; x_k, H_k) \right\} \geq 0.
\]

Combining the two previous inequalities with the definition of \( k'' \) yields \( \lim_{k \to \infty} f(x_k) = -\infty \), which contradicts the fact that \( f \) is bounded as a consequence of Assumptions 4.1 and 4.2. This proves the result. \( \Box \)
We now show that feasible limit points are also first-order solutions of the penalty function.

**Lemma 4.13.** Suppose that Assumptions 4.1–4.4, (4.11), and (4.23) hold. If \( x_* = \lim_{x \in S} x_k \) for some subsequence \( S \) and \( v(x_*) = 0 \), then \( \lim_{x \in S} \Delta q^\phi(s_k^p; x_k, B_k, \bar{\sigma}) = 0 \).

**Proof.** Suppose that there exists a constant \( \epsilon'' > 0 \) and an infinite subsequence
\[
S'' := \{ k \in S : k \geq k'' : \Delta q^\phi(s_k^p; x_k, B_k, \bar{\sigma}) \geq \epsilon'' \},
\]
where \( k'' \) is defined in (4.23). It follows from line 34 of Algorithm 1, (4.11), and (4.23) that
\[
\Delta q^\phi(s_k; x_k, B_k, \bar{\sigma}) \geq \eta \Delta q^\phi(s_k^p; x_k, B_k, \bar{\sigma}) \geq \eta \epsilon'' \quad \text{for} \quad k \in S''.
\]
From (2.6) and (2.16), we know that \( v(x_k) \geq \Delta \ell^\ell(s_k; x_k) \geq \eta \epsilon'' \geq 0 \) for all \( k \), which may be combined with \( \lim_{x \in S} v(x_k) = 0 \) (holds by assumption) to conclude that
\[
\Delta \ell^\ell(s_k; x_k) \leq \frac{\eta \epsilon''}{\sigma + \gamma_v} \quad \text{for} \quad k \in S \text{ sufficiently large},
\]
where \( \gamma_v \in (0, 1) \) is defined in (2.31a). It follows from (2.8), (2.9), (4.24), \( B_k > 0 \), and (4.25) that
\[
\Delta \ell^f(s_k; x_k) \\
\geq \frac{1}{2} s_k^T B_k s_k - \bar{\sigma} \Delta \ell^\ell(s_k; x_k) + \eta \epsilon'' > \eta \epsilon'' - \bar{\sigma} \Delta \ell^\ell(s_k; x_k) \\
\geq \eta \epsilon'' - \bar{\sigma} \frac{\eta \epsilon''}{\sigma + \gamma_v} = \frac{\gamma_v \eta \epsilon''}{\sigma + \gamma_v} =: \epsilon > 0 \quad \text{for} \quad k \in S'' \text{ sufficiently large}.
\]
Combining this with part (iii) of Lemma 4.8, we know that there exists some \( \alpha_f > 0 \) such that \( (\alpha, s_k) \) satisfies (2.31b) for all \( k \in S'' \) sufficiently large and \( \alpha \in (0, \alpha_f] \), since by assumption \( S_0 = S_1 \cup S_2 \cup S_3 \) for \( k \geq k'' \).

Next, we define
\[
\Phi_k := \min_{(v_i, f_i) \in F_k} \left\{ \max \left[ v_i - \alpha_i \eta \Delta \ell^\ell(s_i^p; x_i), \beta v_i \right] \right\} > 0,
\]
where \( F_k \) is the \( k \)th filter. The fact that \( \Phi_k > 0 \) follows since \( v_i > 0 \) for all \( (v_i, f_i) \in F_k \) as a consequence of Lemma 3.1. Moreover, it follows from (4.23) that \( \Phi_k \equiv \Phi_{k''} \) for all \( k \in S'' \) so that \( \Phi_k \equiv \Phi_{k''} > 0 \) for all \( k \in S'' \). Now, pick \( \epsilon'' > 0 \) such that \( \Phi_{k''} - C_v M_s^2 \leq \epsilon'' < \Phi_{k''} \) and consider \( \alpha \) such that
\[
0 < \alpha \leq \frac{\Phi_{k''} - \epsilon''}{C_v M_s^2} \leq 1.
\]
It then follows from Lemma 4.3, the fact that \( \lim_{x \in S} v(x_k) = 0 \), (2.16), (4.23), and (4.28) that
\[
v(x_k + \alpha s_k) \\
\leq v(x_k) - \alpha \Delta \ell^\ell(s_k^p; x_k) + \frac{\alpha^2 C_v}{2} \| s_k \|^2 \leq \epsilon'' + \alpha^2 C_v M_s^2 \\
\leq \epsilon'' + \alpha C_v M_s^2 \leq \epsilon'' + \frac{\Phi_{k''} - \epsilon''}{C_v M_s^2} C_v M_s^2 = \Phi_{k''} \quad \text{for all} \quad k \in S'' \text{ sufficiently large}.
\]
Thus, \( x_k + \alpha s_k \) is acceptable to \( \mathcal{F}_k \equiv \mathcal{F}_{k'} \) for all \( \alpha \) satisfying (4.28) and \( k \in S'' \) sufficiently large.

Combining the above, (4.23), and the structure of Algorithm 1, we conclude that

\[
(4.29) \quad \alpha_k \geq \min \left\{ \frac{\Phi_{k'} - \varepsilon}{C_e M_s^2}, \xi \alpha_f, 1 \right\} =: \alpha_{min} > 0 \quad \text{for all } k \in S'' \text{ sufficiently large,}
\]

where \( \xi \in (0, 1) \) is the backtracking parameter used in Algorithm 1. It then follows from (4.23), (2.31b), (2.16), Lemma 4.5, (2.31a), (4.29), and (4.26) that

\[
(4.30) \quad f(x_k) - f(x_k + \alpha_k \hat{s}_k) \geq \gamma_f \alpha_k \rho_k' \geq \gamma_f \alpha_k \min \left[ C_p \Delta \ell_f (s_k; x_k)^2, \frac{1}{2} \Delta \ell_f (s_k; x_k) \right] \geq \gamma_f \alpha_{min} \min \left[ C_p (\ell_f')^2, \frac{1}{2} \ell_f \right] > 0 \quad \text{for all } k \in S'' \text{ sufficiently large.}
\]

However, for all \( k \in S_o \), it follows from (2.31b), (2.31a), (2.16), and (2.27) that \( f(x_k) - f(x_k + \alpha_k \hat{s}_k) \geq 0 \). This observation combined with (4.30) implies that \( \lim_{k \to \infty} f(x_k) = -\infty \), which contradicts the fact that \( f \) is bounded as a consequence of Assumptions 4.1 and 4.2. This completes the proof.

We now show that limit points are either infeasible stationary points or KKT points for problem (1.1).

**Theorem 4.14.** Suppose that Assumptions 4.1–4.4, (4.11), and (4.23) hold. If \( x_* \) is a limit point of \( \{x_k\} \), then either

(i) \( x_* \) is an infeasible stationary point or

(ii) \( x_* \) is a KKT point for problem (1.1).

**Proof.** Suppose that \( \lim_{k \in S} x_k = x_* \) for some subsequence \( S \). It follows from Lemma 4.12 that \( \lim_{k \to \infty} \Delta \ell^v (s_k^x; x_k) = 0 \), so that if \( v(x_*) > 0 \), then \( x_* \) is an infeasible stationary point (see Definition 2.1). Otherwise, we have that \( v(x_*) = 0 \). In this case, it follows from Lemma 4.13 and (4.11) that \( \lim_{k \in S} \Delta \sigma^v (s_k^x; x_k, B_k, \bar{\sigma}) = 0 \). It follows from this fact, \( v(x_*) = 0 \), and Lemma 2.2 that \( x_* \) is a KKT point for problem (1.1). \( \square \)

**Case 3:** \( |S_o \cup S_b| = \infty \). The next result shows that if \( P\text{-mode} = \text{false} \) at the beginning of the \( k \)th iteration, then \( x_k \) is acceptable to the filter \( \mathcal{F}_k \).

**Lemma 4.15.** If \( P\text{-mode} = \text{false} \) at the beginning of iteration \( k \), then \( x_k \) is acceptable to \( \mathcal{F}_k \).

**Proof.** The result immediately follows from the construction of Algorithm 1 and consideration of the possible outcomes associated with iteration \( k - 1 \). \( \square \)

We first show that the feasibility measure converges to zero along \( S_o \cup S_b \).

**Lemma 4.16.** If Assumptions 4.1–4.3 hold and \( |S_o \cup S_b| = \infty \), then \( \lim_{k \in S_o \cup S_b} \Delta \ell^v (s_k^x; x_k) = 0 \).

**Proof.** To reach a contradiction, suppose that we have the infinite subsequence

\[
S := \{ k \in S_o \cup S_b : \Delta \ell^v (s_k^x; x_k) \geq \epsilon \}
\]

for some constant \( \epsilon > 0 \). It follows from the definition of \( S \), Lemma 4.15, and (2.29) that

\[
(4.31) \quad v_k \leq \max \{ v_j - \alpha_j \eta_j \Delta \ell^v (s_j^x; x_j), \, \beta v_j \} \quad \text{or} \quad f_k \leq f_j - \gamma \min \{ v_j - \alpha_j \eta_j \Delta \ell^v (s_j^x; x_j), \, \beta v_j \}
\]

for \( k \in S \) and \((v_j, f_j) \in \mathcal{F}_k\); note that by construction \((v_k, f_k) \in \mathcal{F}_{k+1} \) for all \( k \in S \). Moreover, it follows from the definitions of \( \Delta \ell^v \) and \( S \) that \( v_k \geq \Delta \ell^v (s_k^x; x_k) \geq \epsilon \) for \( k \in S \). Using Assumptions 4.1 and 4.2 we have a subsequence \( S' \subseteq S \) so that
\[
\lim_{k \in S'} \Delta \ell^v(s_k^*; x_k) = \theta_v \quad \text{and} \quad \lim_{k \in S'} v_k = \theta_v \quad \text{for constants } \theta_v \geq \theta_v \geq \epsilon > 0.
\]

For any \( \epsilon_v \leq (0, \theta_v) \) and \( \epsilon_v \in (0, \theta_v) \), it follows that

\[
\Delta \ell^v(s_k^*; x_k) - |\epsilon_v| < \epsilon_v \quad \text{and} \quad |v_k - \theta_v| < \epsilon_v \quad \text{for all } k \in S' \subseteq S \text{ sufficiently large}.
\]

Using (4.32), the definitions of \( \epsilon_v, \eta_v, \Delta \ell^v \) and \( S, \alpha_k \in (0, 1], S' \subseteq S \), and part (ii) of Lemma 4.8 gives

\[
0 \leq v_k - \alpha_k \eta_v \Delta \ell^v(s_k^*; x_k) < v_k - \alpha_v \eta_v (\theta_v - \epsilon_v) \leq \beta_2 v_k \quad \text{for all } k \in S' \text{ sufficiently large}
\]

and some \( \alpha_v > 0 \), where

\[
\beta_2 := \frac{(\theta_v + \epsilon_v) - \alpha_v \eta_v (\theta_v - \epsilon_v)}{(\theta_v + \epsilon_v)} \in (0, 1)
\]

and \( \beta_2 \) may be forced to lie in \((0, 1)\) by choosing \( \epsilon_v \) sufficiently close to zero and \( \epsilon_v \) sufficiently close to \( \theta_v \). Now define \( \beta^* := \max\{\beta_2, \beta\} \in (0, 1) \)

\[
\epsilon^* = \min \left\{ \frac{1 - \beta^*}{1 + \beta^*}, \epsilon_v \right\} > 0,
\]

and the subsequence \( S'' = \{ k \in S' : |v_k - \theta_v| < \epsilon^* \} \) so that

\[
\frac{2 \beta^*}{1 + \beta^*} \theta_v < v_k < \frac{2}{1 + \beta^*} \theta_v \quad \text{for all } k \in S'' \text{ sufficiently large}.
\]

Given \( k \in S'' \), define \( k^+ \in S'' \) to be the successor to \( k \) in \( S'' \). It then follows from (4.34), the definition of \( \beta^* \), and (4.33) that

\[
v_{k^+} > \frac{2 \beta^*}{1 + \beta^*} \theta_v > \beta^* v_k = \max\{\beta_2, \beta\} v_k
\]

\[
\geq \max \left\{ v_k - \alpha_k \eta_v \Delta \ell^v(s_k^*; x_k), \beta v_k \right\} \quad \text{for all } k \in S''.
\]

Since \( S'' \subseteq S' \subseteq S \), it follows from the previous inequality, the definition of \( k^+ \), the fact that \( (v_k, f_k) \in F_{k+}, (4.31) \), the definition of \( \Delta \ell^v(s_k^*; x_k) \), \( \alpha_k \in (0, 1], \eta_v \in (0, 1), \beta \in (0, 1), \gamma \in (0, 1), \theta > \epsilon_v \geq \epsilon^*, \) and the definition of \( S'' \) that

\[
f_k - f_{k^+} \geq \gamma \min \left\{ v_k - \alpha_k \eta_v \Delta \ell^v(s_k^*; x_k), \beta v_k \right\}
\]

\[
= \gamma \min \left\{ (1 - \alpha_k \eta_v)v_k + \alpha_k \eta_v \|v(x_k) + J(x_k) s_k^*\|_1, \beta v_k \right\}
\]

\[
\geq \gamma \min \{1 - \alpha_k \eta_v, \beta\} v_k \geq \gamma \min \{1 - \eta_v, \beta\}(\theta_v - \epsilon^*) > 0 \quad \text{for all } k \in S''.
\]

Summing over \( k \in S'' \), we deduce that \( \lim_{k \in S''} f(x_k) = -\infty \), which contradicts Assumptions 4.1 and 4.2.

We now prove that our optimality measure for \( \phi \) converges to zero along a certain subsequence.

Lemmas 4.17. Suppose that Assumptions 4.1–4.4 and (4.11) hold and that \( |S_v \cup S_b| = \infty \).

(i) If \( |S_v| = \infty \) and \( \lim_{k \in S_v} x_k = x_* \) for some \( x_* \) satisfying \( v(x_*) = 0 \), then

\[
\lim_{k \in S_v} \Delta q^v(s_k^*; x_k, B_k, \bar{a}) = 0.
\]
(ii) If $|S_v| < \infty$ and $\lim_{k \in S_v} x_k = x_*$ for some $x_*$ satisfying $v(x_*) = 0$, then

$$\liminf_{k \in S_v} \Delta q^\theta(s_k^p; x_k, B_k, \tilde{\sigma}) = 0.$$

Proof. We first prove part (i). To obtain a contradiction, suppose that there exists the subsequence

$$S' := \{ k \in S_v : k \geq k' \text{ and } \Delta q^\theta(s_k^p; x_k, B_k, \tilde{\sigma}) \geq \epsilon' \}$$

for some constant $\epsilon' > 0$ and $k'$ defined in (4.11). It then follows from line 34 of Algorithm 1 that

$$\Delta q^\theta(s_k; x_k, B_k, \tilde{\sigma}) \geq \eta_0 \Delta q^\theta(s_k^p; x_k, B_k, \tilde{\sigma}) \geq \eta_0 \epsilon' \text{ for } k \in S'.$$

Then, since $v(x_*) = 0$ by assumption, we may use (4.35) (analogous to (4.24)) and follow the same steps that led to (4.26) to show that

$$\Delta f^\ell(s_k; x_k) \geq \epsilon' \geq \gamma_v \Delta f^\ell(s_k; x_k) \text{ for } k' \in S' \text{ sufficiently large and some } \epsilon' > 0,$$

where the second inequality follows from $\lim_{k \in S_v} x_k = x_*$, $v(x_*) = 0$, and the definition of $\Delta f^\ell$. Thus, (2.30) does not hold, which implies that $k' \notin S_v$. This is a contradiction and proves part (i).

We now prove part (ii), where $|S_v| < \infty = |S_b|$. To obtain a contradiction, suppose that

$$\Delta q^\theta(s_k^p; x_k, B_k, \tilde{\sigma}) \geq \epsilon' \text{ for } k \in S_b \text{ sufficiently large}$$

and some constant $\epsilon' > 0$. It then follows from line 34 of Algorithm 1 that

$$\Delta q^\theta(s_k; x_k, B_k, \tilde{\sigma}) \geq \eta_0 \Delta q^\theta(s_k^p; x_k, B_k, \tilde{\sigma}) \geq \eta_0 \epsilon' \text{ for } k \in S_b \text{ sufficiently large}.$$

Since (4.36) is analogous to (4.35), we may again conclude as above that

$$\Delta f^\ell(s_k; x_k) \geq \epsilon' \geq \gamma_v \Delta f^\ell(s_k; x_k) \text{ for } k \in S_b \text{ sufficiently large and some } \epsilon' > 0.$$

Using (4.37), $\lim_{k \in S_b} v(x_k) = v(x_*) = 0$, and part (iii) of Lemma 4.8, we may conclude that there exists $\alpha_f > 0$ such that $(\alpha, s_k)$ satisfies (2.31b) for all $\alpha \in (0, \alpha_f]$ and $k \in S_b$ sufficiently large. Now, if $\alpha_k \to 0$ along some subsequence $S_{b}^* \subseteq S_b$, then it follows from the previous sentence and (4.37) that $(\alpha_k, s_k)$ satisfies (2.31a) and (2.31b) for all $k \in S_b$ sufficiently large. We now show that $x_k + \alpha_k s_k$ is also acceptable to the filter $\mathcal{F}_k$ for all $k \in S_{b}^*$ sufficiently large.

To this end, let $(v_i, f_i) \in \mathcal{F}_k$ for some $k \in S_{b}^*$. It then follows from Lemma 4.15 that either $v_k \leq \max \{ v_i - \alpha_i \eta_0 \Delta f^\ell(s_k^p; x_k), \beta v_i \}$ or $f_k \leq f_i - \gamma \min \{ v_i - \alpha_i \eta_0 \Delta f^\ell(s_k^p; x_k), \beta v_i \}$. In this first case, it follows from the definition of a b-pair that $v(x_k + \alpha_k s_k) \leq v_k \leq \max \{ v_i - \alpha_i \eta_0 \Delta f^\ell(s_k^p; x_k), \beta v_i \}$ for all $k \in S_b$. In the second case, we have from the fact that (2.31b) holds for $k \in S_{b}^*$ sufficiently large (recall that $\alpha_k \to 0$ on $S_{b}^*$), (4.37), and Lemma 4.5 that $f(x_k + \alpha_k s_k) \leq f_k \leq f_i - \gamma \min \{ v_i - \alpha_i \eta_0 \Delta f^\ell(s_k^p; x_k), \beta v_i \}$. Thus, in either case we have that $(v(x_k + \alpha_k s_k), f(x_k + \alpha_k s_k))$ is acceptable to the single element filter $(v_i, f_i)$ for all $k \in S_{b}^*$ sufficiently large. Since this element $(v_i, f_i)$ of the filter $\mathcal{F}_k$ was arbitrary, we may conclude that $(v(x_k + \alpha_k s_k), f(x_k + \alpha_k s_k))$ is, in fact, acceptable to the filter $\mathcal{F}_k$ for all $k \in S_{b}^*$ sufficiently large.
To summarize, we have shown that \((\alpha_k, s_k)\) is an \(o\)-pair for \(k \in \mathcal{S}_b\) sufficiently large. This is a contradiction since Algorithm 1 would have labeled such an iterate as an \(o\)-iterate, not a \(b\)-iterate. Thus, there exists \(\alpha_b\) such that \(\alpha_k \geq \alpha_b > 0\) for all \(k \in \mathcal{S}_b\) sufficiently large. Combining this with (2.1), (2.34), \(v(\cdot) \geq 0\), Lemma 4.7, (2.8), (2.16), and (4.37) gives

\[
(4.38) \quad f(x_k) - f(x_k + \alpha_k \delta_k) = \phi(x_k; \bar{\sigma}) - \phi(x_k + \alpha_k \delta_k; \bar{\sigma}) - \bar{\sigma}(v(x_k) - v(x_k + \alpha_k \delta)) = \sum_{i = k+1}^{k^+} \sum_{i = k+1}^{k^+} \gamma_o \alpha_i \rho_i^\phi \geq \sum_{i = k+1}^{k^+} \gamma_o \alpha_i \rho_i^\phi \geq 0 \quad \text{for } k \in \mathcal{S}_b \text{ sufficiently large,}
\]

which may be combined with (2.1), \(v(\cdot) \geq 0\), and (2.33) to conclude that

\[
(4.39) \quad f(x_{k+1}) - f(x_k) \geq 0, \quad f(x_{k+1}) - f(x_{k+1}) \geq 0, \quad f(x_{k+1}) - f(x_{k+1}) \geq 0, \quad f(x_{k+1}) - f(x_{k+1}) \geq 0.
\]

It then follows from (4.38) and (4.39) that

\[
(4.40) \quad f(x_k) - f(x_{k+1}) = (f(x_k) - f(x_k + \alpha_k \delta_k)) + (f(x_{k+1}) - f(x_k)) > \gamma_o \alpha_b \min \left[ C_\rho (\epsilon^f)^2, \frac{1}{2} \epsilon^f \right] - 2\bar{\sigma} v(x_k) \quad \text{for } k \in \mathcal{S}_b \text{ sufficiently large.}
\]

Next, since \(\lim_{k \in \mathcal{S}_b} v(x_k) = 0\) we know that

\[
v(x_k) \leq \frac{1}{\gamma_o} \gamma_o \alpha_b \min \left[ C_\rho (\epsilon^f)^2, \frac{1}{2} \epsilon^f \right] \quad \text{for } k \in \mathcal{S}_b \text{ sufficiently large,}
\]

which may be combined with (4.40) to deduce that

\[
(4.41) \quad f(x_k) - f(x_{k+1}) > \frac{1}{2} \gamma_o \alpha_b \min \left[ C_\rho (\epsilon^f)^2, \frac{1}{2} \epsilon^f \right] =: \epsilon^o > 0 \quad \text{for } k \in \mathcal{S}_b \text{ sufficiently large.}
\]
If we define \( \hat{k}^+ \) to be the first \( b \)-iteration greater than \( k \) (thus, \( \hat{k}^+ \geq k^+ \)), it follows from (4.41), the fact that Algorithm 1 does not allow further \( p \)-iterations until it has its next \( b \)-iteration, and the fact that the objective \( f \) is decreased during \( o \)-iterations that \( f(x_k) - f(x_{k^+}) > \epsilon^o \) for \( k \in S_b \) sufficiently large. Since \( |S_b| = \infty \), this implies that \( \lim_{k \in S_b} f(x_k) = -\infty \), which contradicts the fact that \( f \) is bounded as a consequence of Assumptions 4.1 and 4.2. This proves the result. \( \square \)

We now show that limit points of \( \{x_k\}_{S_r \cup S_b} \) are infeasible stationary or KKT point for problem (1.1).

**Theorem 4.18.** Suppose that the Assumptions 4.1–4.4, (4.11), and \( |S_r \cup S_b| = \infty \) hold. Then, there exists a limit point \( x_\ast \) of \( \{x_k\}_{S_r \cup S_b} \) such that either

(i) \( x_\ast \) is a KKT point of problem (1.1) or

(ii) \( x_\ast \) is an infeasible stationary point.

**Proof.** From Assumptions 4.1 and 4.2 we know that there exists a limit point \( x_\ast \) of \( \{x_k\}_{S_r \cup S_b} \). First, if \( v(x_\ast) > 0 \), then it follows from Lemma 4.16 and Lemma 2.1 that \( x_\ast \) is an infeasible stationary point, which is part (ii) of this theorem. Second, if \( v(x_\ast) = 0 \) and \( |S_b| = \infty \), then it follows from part (i) of Lemma 4.17 and Lemma 2.2 that \( x_\ast \) is a KKT point of problem (1.1). This is case (i) of this theorem. Finally, if \( v(x_\ast) = 0 \) and \( |S_b| < \infty \) (so that \( |S_b| = \infty \)), then it follows from part (ii) of Lemma 4.17 and Lemma 2.2 that \( x_\ast \) is a KKT point of problem (1.1), which once again is case (i) of the theorem. \( \square \)

### 4.2. Convergence analysis under unbounded weighting parameter.

We now consider the situation when the weighting parameter increases without bound, i.e., that

\[
\lim_{k \to \infty} \sigma_k = \infty.
\]

Our analysis begins with the following lemma, which is similar to [9, Lemma 3.8].

**Lemma 4.19.** Suppose that Assumptions 4.1–4.4 are satisfied, (4.42) holds, \( x_\ast \) is a limit point of \( \{x_k\} \) satisfying \( v(x_\ast) > 0 \), and \( \Delta f'(s_k^\ast; x_\ast) > 0 \), where \( s_k^\ast \) is the solution to

\[
\min_{(s,r) \in \mathbb{R}^{n+m}} e^T r \quad \text{subject to} \quad c(x_\ast) + J(x_\ast) s + r \geq 0, \quad r \geq 0, \quad \|s\|_{\infty} \leq \delta,
\]

for some \( \delta \in [\delta_{\min}, \delta_{\max}] \). Then, along any subsequence \( \{x_k\}_{k \in K} \) that converges to \( x_\ast \), the weighting parameter is updated only a finite number of times.

**Proof.** We begin by defining

\[
s_k^\sigma(\sigma) := \arg\min_{s \in \mathbb{R}^n} q^\sigma(s; x_k, B_k, \sigma)
\]

and

\[
\mu := \mu(\sigma) := \left(1 - \frac{\eta_\sigma}{\eta_v}\right) \sigma < \sigma,
\]

where we used the fact that \( 0 < 1 - \eta_\sigma/\eta_v < 1 \) holds since \( 0 < \eta_\sigma < \eta_v < 1 \) is defined in (2.19).

Using the fact that \( \Delta q^\sigma(s_k^\sigma(\mu); x_k, B_k, \mu) \geq 0 \), (2.9), and the definition of \( \mu = \mu(\sigma) \), we can see that
Moreover, since the Newton step 

which implies that

Next, it follows from (2.10), the choice

Moreover, since the Newton step 

Since \( \Delta^{\ell'}(s_{k}^*; x_k) > 0 \) and \( \lim_{k \to K} x_k = x^* \) by assumption, it follows from [17, Theorem 3.2.8] that there exists \( \epsilon \in (0,1) \) and \( k' \) such that

Moreover, since the Newton step \(-B_k^{-1} g_k\) minimizes \( q^f(s; x_k, B_k)\), it follows from Assumption 4.3 that

Next, it follows from (2.10), the choice \( \delta_k \in [\delta_{\min}, \delta_{\max}]\), norm inequalities, and Assumption 4.3 that

Then, (4.47), (4.48), and Assumptions 4.1 and 4.2 imply the existence of a constant \( C_{\alpha} > 0 \) such that

We now define

and the associated infinite subsequence

It follows from the fact that \( \Delta q^\phi(s_{k}^*(\sigma); x_k, B_k, \sigma) \geq \Delta q^\phi(s_{k}^*; x_k, B_k, \sigma) \) (by the definition of \( s_{k}^*(\sigma) \)), (2.9), (4.49), (4.46), and (4.50) that

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We may now use the definition of $S'$, (4.52), the fact that $s_k^p(\sigma_k) \equiv s_k^p$, (2.15), and (2.16) to show that

(4.53)  
\[ \tau_k = 1, \ s_k = s_k^p, \ \text{and} \ \Delta q^p(s_k; x_k, B_k, \sigma_{k+1}) \geq \eta_0 \Delta q^p(s_k^p; x_k, B_k, \sigma_{k+1}) \ \text{for} \ k \in S' \]

since $\eta_0 \in (0, 1)$ in Algorithm 1. Next, it follows from (4.53), (2.9), $B_k \succ 0$, the fact that $s_k^p \equiv s_k^p(\sigma_k)$ and $s_k^p$ minimizes $q^p(s; x_k, B_k, \sigma_k)$, (4.45), the fact that $\mu(\sigma_k) \geq \mu(\sigma_{crit})$ for $k \in S'$, and (4.52) that

(4.54)  
\[ \Delta \ell^p(s_k; x_k, \sigma_k) \geq \Delta q^p(s_k^p; x_k, B_k, \sigma_k) \geq \Delta q^p(s_k^p(\mu(\sigma_k)); x_k, B_k, \sigma_k) \]
\[ \geq \frac{\eta}{\eta_e} \sigma_k \Delta \ell^p(s_k^p(\mu(\sigma_k)); x_k) \]
\[ \geq \frac{\eta}{\eta_e} \sigma_k \eta \Delta v^p(s_k^p; x_k) \]

We now conclude from (4.53), (4.54), (2.19), and the fact that the weighting parameter is only increased in lines 12 and 34 of Algorithm 1 that $\sigma_k$ is increased a finite number of times on $K$.

We now consider feasible limit points at which the MFCQ [35] holds.

**Lemma 4.20.** Suppose that Assumptions 4.1–4.4 are satisfied, (4.42) holds, $x_*$ is a limit point of $\{x_k\}$ at which $v(x_*) = 0$, and the MFCQ holds. Then, the following hold for all $x_k$ sufficiently close to $x_*$ and $\sigma_k$ sufficiently large: (i) $\ell^p(s_k^p; x_k) = v(x_k)$; (ii) $s_k = s_k^p$; and (iii) $\sigma_k$ is not increased during iteration $k$.

**Proof.** We may use [9, Lemmas 3.12 and 3.13] since the proofs only used the properties of the MFCQ, the continuity of the problem functions $f$ and $g$, and the convexity of their penalty and steering subproblems. Their subproblem [9, equations 2.7(a–d)] is equivalent to our predictor subproblem (2.12) and both methods minimize the same quadratic model of the penalty function. A small difference is that our predictor subproblem is designed so that if $\ell^p(s_k^p; x_k) = 0$, then $\ell^p(s_k^p; x_k) = 0$ as well; they satisfy this requirement by increasing their penalty parameter in Step 4a of [9, equation 2.11] and re-solving for a new step. Their steering subproblem [9, equations 2.9(a–c)] is equivalent to (2.10).

The assumptions of this lemma and [9, Lemma 3.12] imply the existence of $r > 0$ and $k' \geq 0$ so that

(4.55)  
\[ \ell^p(s_k^p; x_k) = v(x_k) \]

where $S' := \{k : \|x_k - x_*\| \leq r \text{ and } k \geq k'\}$, which proves part (i). The inequality $\Delta \ell^p(s_k^p; x_k) \geq 0$, (4.55), and the definition of $\Delta \ell^p$ imply

\[ \Delta \ell^p(s_k^p; x_k) \geq v(x_k) - \ell^p(s_k^p; x_k) = \Delta \ell^p(s_k^p; x_k) \geq \eta_r \Delta v^p(s_k^p; x_k) \]

where $\eta_r \in (0, 1)$ is defined in (2.16). Thus, we conclude from (2.16) that $\tau_k = 1$ and $s_k = s_k^p$ for $k \in S'$, which proves part (ii). Finally, it follows from [9, Lemma 3.13] and the assumptions of this lemma that

(4.56)  
\[ \Delta q^p(s_k^p; x_k, B_k, \sigma_k) \geq \sigma_k \eta \sigma v(x_k) \geq \sigma_k \eta \sigma \Delta \ell^p(s_k^p; x_k) \]

where the last inequality follows from the definition of $\Delta \ell^p$. It then follows from part (ii) of this lemma, (2.9), $B_k \succ 0$, and (4.56) that

\[ \Delta \ell^p(s_k^p; x_k, \sigma_k) = \Delta \ell^p(s_k^p; x_k, \sigma_k) \geq \Delta q^p(s_k^p; x_k, B_k, \sigma_k) \geq \sigma_k \eta \sigma \Delta \ell^p(s_k^p; x_k) \]
We may conclude from this inequality, (2.19), and the fact that \( \sigma_k \) will not be increased on line 34 as a result of part (ii) of this lemma that \( \sigma_{k+1} = \sigma_k \) for \( k \in S' \), which proves part (iii).

**Theorem 4.21.** If Assumptions 4.1–4.4 and (4.42) hold, there is a limit point \( x_* \) such that either

(i) \( x_* \) is an infeasible stationary point or 
(ii) \( x_* \) is feasible, but the MFCQ does not hold.

**Proof.** Let \( D \) to be the infinite index set consisting of the iterations for which the weighting parameter is increased. Then, let \( x_* \) be a limit point of \( \{x_k\}_{k \in D} \), which must exist as a consequence of Assumptions 4.1 and 4.2. First, suppose that \( v(x_*) > 0 \). It then follows from Lemma 4.19 that if \( \Delta^\ell v(s_*^*; x_*) > 0 \) (\( s_*^* \) is defined in Lemma 4.19), then the weighting parameter is updated only a finite number of times along \( D \), which is a contradiction. Therefore, we deduce that \( \Delta^\ell v(s_*^*; x_*) = 0 \) and consequently that \( x_* \) is an infeasible stationary point. Second, suppose that \( v(x_*) = 0 \). It then follows from Lemma 4.20 that if the MFCQ holds at \( x_* \), then \( \sigma_k \) will be increased only a finite number of times along \( D \). This is a contradiction and, therefore, the MFCQ does not hold at \( x_* \). \( \square \)

5. Conclusions. In this paper, we presented a new filter line search method that replaced the traditional restoration phase with a penalty mode that systematically decreased an exact penalty function. Importantly, we solved a single strictly convex quadratic program subproblem during each iteration that was always feasible. Each search direction was defined as a convex combination of a steering step (a solution of a linear program) that represented the best local improvement in constraint violation and a predictor step that reduced our strictly convex quadratic model of the exact penalty function. We also allowed for the computation of an accelerator step defined as a solution to a simple equality constrained quadratic program (plus trust-region constraint) to promote fast local convergence. In this manner, the trial step always incorporated information from both the objective function and constraint violation. To further promote step acceptance, we utilized second-order information in the computation of Cauchy steps that provided realistic measurements of the decrease one might expect from the nonlinear problem functions. By using local feasibility estimates that emerged during the steering process, we defined a new and improved margin (envelope) of the filter. This new definition encouraged the acceptance of steps that made reasonable progress but might be considered inadmissible by a traditional filter. Under standard assumptions, we proved global convergence of our algorithm.

The fact that every subproblem of our method is feasible has an interesting (favorable) consequence when compared to previous SQP filter methods. Those methods trigger a restoration phase in multiple situations, the most common being when the traditional SQP subproblem is infeasible. In this case, the primary role of the restoration phase is to obtain a new feasible subproblem. This undesirable situation is not encountered in our method since all subproblems are feasible. Our method still may enter a penalty phase, but only when overwhelming evidence indicates that previously added filter entries are blocking progress. We believe this feature of our method is far more attractive and practical in comparison to previous filter methods.

Local convergence issues have not been considered in this paper and are currently under investigation. It is evident that our method—like filter methods and SQP methods based on exact penalty functions—may experience the Maratos effect [36], i.e., reject the unit step (the traditional SQP step) when the current iterate is arbitrarily...
close to a minimizer. We remain optimistic, however, that superlinear convergence may be established if we make common second-order optimality assumptions on the minimizer and include either a second-order correction strategy or a nonmonotone approach [17, section 15.3.2.3].

Simple and straightforward modifications to our method allow for the solution of problems defined by a mixture of inequality and equality constraints. For instance, the definition of the constraint violation would be augmented to represent an $\ell_1$ measure of infeasibility for both inequality and equality constraints. Each key subproblem must also be modified. For example, (2.13) would additionally include the linearized equality constraints augmented by a pair of nonnegative elastic variables. Otherwise, the algorithm remains unchanged.

We are currently implementing our algorithm. Once a robust and well-tested code is obtained, we will investigate many interesting practical questions related to our method. These questions include (i) computing the frequency with which our method enters a penalty mode relative to a traditional restoration phase; (ii) evaluating the benefits and possible disadvantages of our penalty mode versus a traditional restoration phase on test problems that trigger such phases; (iii) studying the stability and quality of the iterations typically generated by a traditional restoration phase and our penalty mode; and (iv) investigating the practical benefits of defining our filter margin adaptively based on local estimates. However, such tests should only be performed after a polished implementation is obtained so that reliable conclusions may be obtained.

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A FILTER METHOD WITH UNIFIED STEP COMPUTATION


