SOLVING THE TRUST-REGION SUBPROBLEM USING THE
LANCZOS METHOD∗

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Abstract. The approximate minimization of a quadratic function within an ellipsoidal trust
region is an important subproblem for many nonlinear programming methods. When the number of
variables is large, the most widely used strategy is to trace the path of conjugate gradient iterates
either to convergence or until it reaches the trust-region boundary. In this paper, we investigate
ways of continuing the process once the boundary has been encountered. The key is to observe that
the trust-region problem within the currently generated Krylov subspace has a very special structure
which enables it to be solved very efficiently. We compare the new strategy with existing methods.
The resulting software package is available as HSL_VF05 within the Harwell Subroutine Library.

Key words. trust-region subproblem, Lanczos method, conjugate gradients, preconditioning

AMS subject classifications. 90C20, 90C30, 65K05, 65F10

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1. Introduction. Trust-region methods for unconstrained minimization are
blessed with both strong theoretical convergence properties and a good reputation
in practice. The main computational step in these methods is to find an approximate
minimizer of some model of the true objective function within a “trust” region for
which a suitable norm of the correction lies within a given bound. This restriction is
known as the trust-region constraint, and the bound on the norm is its radius. The
radius is adjusted so that successive model problems mimic the true objective within
the trust region.

The most widely used models are quadratic approximations to the objective func-
tion, as these are simple to manipulate and may lead to rapid convergence of the
underlying method. From a theoretical point of view, the norm which defines the
trust region is irrelevant so long as it is “uniformly” related to the ℓ2-norm. From a
practical perspective, this choice certainly affects the subproblem and thus the meth-
ods one can consider when solving it. The most popular practical choices are the ℓ2-
and ℓ∞-norms and weighted variants thereof. In our opinion, it is important that the
choice of norm reflects the underlying geometry of the problem; simply picking the
ℓ2-norm may not be adequate when the problem is large and the eigenvalues of the
Hessian of the model widely spread. We believe that weighting the norm is essential
for many large-scale problems.

In this paper, we consider the solution of the quadratic-model trust-region sub-
problem in a weighted ℓ2-norm. We are interested in solving large problems and thus
cannot rely solely on factorizations of the matrices involved. We thus concentrate on

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iterative methods. If the model of the Hessian is known to be positive definite and the trust-region radius sufficiently large that the trust region constraint is inactive at the unconstrained minimizer of the model, the obvious way to solve the problem is to use the preconditioned conjugate-gradient method. Note that the role of the preconditioner here is the same as the role of the norm used for the trust-region, namely, to change the underlying geometry so that the Hessian in the rescaled space is better conditioned. Thus, it will come as no surprise that the two should be intimately connected. Formally, we shall require that the weighting in the $\ell_2$-norm and the preconditioning be performed by the same matrix.

When the radius is smaller than a critical value, the unconstrained minimizer of the model will no longer lie within the trust region and thus the required solution will lie on the trust-region boundary. The simplest strategy in this case is to consider the piecewise linear path connecting the conjugate-gradient iterates and to stop at the point where this path leaves the trust region. Such a strategy was first proposed independently by Steihaug [22] and Toint [23], and we shall refer to the terminating point as the Steihaug–Toint point. Remarkably, it is easy to establish the global convergence of a trust-region method based on such a simple strategy. The key is that global convergence may be proved provided that the accepted estimate of the solution has a model value no larger than at the Cauchy point (see [14]). The Cauchy point is simply the minimizer of the model within the trust region along the preconditioned steepest-descent direction. As the first segment on the piecewise-linear conjugate-gradient path gives precisely this point, and as the model value is monotonically decreasing along the entire path, the Steihaug–Toint strategy ensures convergence.

If the model Hessian is indefinite, the solution must also lie on the trust-region boundary. This case may also be simply handled using preconditioned conjugate gradients. Once again the piecewise linear path is followed until either it leaves the trust region or a segment with negative curvature is found. (A vector $p$ is a direction of negative curvature if the inner product $\langle p, Hp \rangle < 0$, where $H$ is the model Hessian.) In the latter case, the path is continued downhill along this direction of negative curvature as far as the constraint boundary. This variant was proposed in [22], while [23] suggests simply returning to the Cauchy point. As before, global convergence is ensured at either of these terminating points, as the objective function values there are no larger than at the Cauchy point. For consistency with the previous paragraph, we shall continue to refer to the terminating point in Steihaug's algorithm as the Steihaug–Toint point, although strictly Toint's point in this case may be different.

The Steihaug–Toint method is basically unconcerned with the trust region until it blunders into its boundary and stops. This is rather unfortunate, particularly as considerable experience has shown that this frequently happens during the first few, and often the first, iteration(s) when negative curvature is present. The resulting step is then barely, if at all, better than the Cauchy direction, and this may lead to a slow but globally convergent algorithm in theory and a barely convergent method in practice. In this paper, we consider an alternative which aims to avoid this drawback by trying harder to solve the subproblem when the boundary is encountered, while maintaining the efficiencies of the conjugate-gradient method so long as the iterates lie interior. The mechanism we use is the Lanczos method.

The paper is organized as follows. In section 2 we formally define the problem and any notation that we will use. The basis of our new method is given in section 3, while in section 4, we will review basic properties of the preconditioned conjugate-gradient and Lanczos methods. Our new method is given in detail in section 5. Some
numerical experiments demonstrating the effectiveness of the approach are given in section 6, and a number of conclusions and perspectives are drawn in the final section.

2. The trust-region subproblem and its solution. Let \( M \) be a symmetric positive-definite easily invertible approximation to the symmetric matrix \( H \). Furthermore, define the \( M \)-norm of a vector as

\[
\|s\|_M^2 = \langle s, Ms \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is the usual Euclidean inner product. In this paper, we consider the \( M \)-norm trust-region problem

\[
\text{minimize } q(s) \equiv (g, s) + \frac{1}{2} \langle s, Hs \rangle \text{ subject to } \|s\|_M \leq \Delta
\]

for some vector \( g \) and radius \( \Delta > 0 \).

A global solution to the problem is characterized by the following result.

**Theorem 2.1** (see [4], [20]). Any global minimizer \( s^M \) of \( q(s) \) subject to \( \|s\|_M \leq \Delta \) satisfies the equation

\[
H(\lambda^M)s^M = -g,
\]

where \( H(\lambda^M) \equiv H + \lambda M \) is positive semidefinite, \( \lambda^M \geq 0 \), and \( \lambda^M(\|s^M\|_M - \Delta) = 0 \). If \( H(\lambda^M) \) is positive definite, \( s^M \) is unique.

This result is the basis of a series of related methods for solving the problem which are appropriate when forming factorizations of \( H(\lambda) \equiv H + \lambda M \) for a number of different values of \( \lambda \) is realistic. Then either the solution lies interior, and hence \( \lambda^M = 0 \) and \( s^M = -H^+g \), or the solution lies on the boundary and \( \lambda^M \) satisfies the nonlinear equation

\[
\|H(\lambda^M)g\|_M = \Delta,
\]

where \( H^+ \) denotes the pseudoinverse of \( H \). Equation (2.3) is straightforward to solve using a safeguarded Newton iteration, except in the so-called hard case for which \( g \) lies in the null-space of \( H(\lambda^M) \). In this case, an additional vector in the range-space of \( H(\lambda^M) \) may be required if a solution on the trust-region boundary is sought. Goldfeldt, Quandt, and Trotter [5], Hebden [7], and Gay [4] all proposed algorithms of this form. The most sophisticated algorithm to date, that by Moré and Sorensen [9], is available as subroutine \texttt{GQTPAR} in the \texttt{MINPACK-2} package and guarantees that a nearly optimal solution will be obtained after a finite number of factorizations.

While such algorithms are appropriate for large problems with special Hessian structure—such as for band matrices—the demands of a factorization at each iteration limit their applicability for general large problems. It is for this reason that methods which do not require factorizations are of interest.

Throughout this paper, we shall denote the \( k \) by \( k \) identity matrix by \( I_k \) and its \( j \)th column by \( e_j \). A set of vectors \( \{q_i\} \) are said to be \( M \)-orthonormal if \( \langle q_i, Mq_j \rangle = \delta_{ij} \), the Kronecker delta, and the matrix \( Q_k = (q_0 \cdots q_k) \) formed from these vectors is an \( M \)-orthonormal matrix. The set of vectors \( \{p_i\} \) are \( H \)-conjugate (or \( H \)-orthogonal) if \( \langle p_i, Hp_j \rangle = 0 \) for \( i \neq j \).
A new algorithm for large-scale trust-region subproblems. To set the scene for this paper, we recall that the Cauchy point may be defined as the solution to the problem

\[
\begin{align*}
\text{minimize } & q(s) \equiv \langle g, s \rangle + \frac{1}{2} \langle s, Hs \rangle \quad \text{subject to } \|s\|_M \leq \Delta, \\
\end{align*}
\]

that is, as the minimizer of \(q\) within the trust region where \(s\) is restricted to the one-dimensional subspace \(\text{span} \{ M^{-1}g \} \). The dogleg methods (see [3], [13]) aim to solve the same problem over a particular two-dimensional subspace (a one-dimensional arc), while [19] does the same over a general two-dimensional subspace. In each of these cases the solution is easy to find as the search space is small. The difficulty with the general problem (2.1) is that the search space \(\mathbb{R}^n\) is large. This leads immediately to the possibility of solving a compromise problem

\[
\begin{align*}
\text{minimize } & q(s) \quad \text{subject to } \|s\|_M \leq \Delta, \\
\end{align*}
\]

where \(S\) is a specially chosen subspace of \(\mathbb{R}^n\).

Now consider the Steihaug–Toint algorithm at an iteration \(k\) before the trust-region boundary is encountered. In this case, the point \(s_{k+1}\) is the solution to (3.2) with the set (3.3)

\[
S = K_k \overset{\text{def}}{=} \text{span} \{ M^{-1}g, (M^{-1}H)M^{-1}g, (M^{-1}H)^2M^{-1}g, \ldots, (M^{-1}H)^kM^{-1}g \},
\]

the Krylov space generated by the starting vector \(M^{-1}g\), and matrix \(M^{-1}H\). That is, the Steihaug–Toint algorithm gradually widens the search space using the very efficient preconditioned conjugate-gradient method. However, as soon as the Steihaug–Toint algorithm moves across the trust-region boundary, the terminating point \(s_{k+1}\) no longer necessarily solves the problem (3.2) over the set (3.3); indeed it is very unlikely to do so when \(k > 0\). (As the iterates generated by the method increase in \(M\)-norm, once an iterate leaves the trust region, the solution to (3.2)–(3.3), and thus (2.1), must lie on the boundary. See [22, Theorem 2.1] for details.) Can we do better? Yes, by recalling that the preconditioned conjugate-gradient and Lanczos methods generate different bases for the same Krylov space.

The preconditioned conjugate-gradient and Lanczos methods. The preconditioned conjugate-gradient and Lanczos methods may be viewed as efficient techniques for constructing different bases for the same Krylov space, \(K_k\). The conjugate-gradient method aims for an \(H\)-conjugate basis, while the Lanczos method obtains an \(M\)-orthonormal basis.

**Algorithm 4.1** (the preconditioned conjugate-gradient method). Set \(g_0 = g\), and let \(v_0 = M^{-1}g_0\) and \(p_0 = -v_0\). For \(j = 0, 1, \ldots, k - 1\), perform the iteration

\[
\begin{align*}
\alpha_j &= \langle g_j, v_j \rangle / \langle p_j, Hp_j \rangle, \\
g_{j+1} &= g_j + \alpha_jHp_j, \\
v_{j+1} &= M^{-1}g_{j+1}, \\
\beta_j &= \langle g_{j+1}, v_{j+1} \rangle / \langle g_j, v_j \rangle, \\
p_{j+1} &= -v_{j+1} + \beta_jp_j.
\end{align*}
\]
Algorithm 4.2 (preconditioned Lanczos method). Set $t_0 = g$, $w_{-1} = 0$, and, for $j = 0, 1, \ldots, k$, perform the iteration

$$\begin{align*}
y_j &= M^{-1}t_j, \\
\gamma_j &= \sqrt{\langle t_j, y_j \rangle}, \\
w_j &= t_j/\gamma_j, \\
q_j &= y_j/\gamma_j, \\
\delta_j &= \langle q_j, Hq_j \rangle, \\
t_{j+1} &= Hq_j - \delta_j w_j - \gamma_j w_{j-1}. 
\end{align*}$$

The conjugate-gradient method generates the basis

$$K_k = \text{span}\{p_0, p_1, \ldots, p_k\}$$
from Algorithm 4.1, while the Lanczos method generates the basis

$$K_k = \text{span}\{q_0, q_1, \ldots, q_k\}$$
from Algorithm 4.2. The Lanczos iteration is often written in the more compact form

$$HQ_k - MQ_k T_k = \gamma_{k+1} w_{k+1} \epsilon_{k+1}^T$$
and

$$Q_k^T M Q_k = I_{k+1},$$

where $Q_k$ is the matrix $(q_0 \cdots q_k)$ and the matrix

$$T_k = \begin{pmatrix}
\delta_0 & \gamma_1 \\
\gamma_1 & \delta_1 \\
& \ddots \\
& & \delta_{k-1} & \gamma_k \\
& & & \delta_k
\end{pmatrix}$$
is tridiagonal. It then follows directly that

$$Q_k^T H Q_k = T_k,$$
$$Q_k^T g = \gamma_0 \epsilon_1,$$
and

$$g = M y_0 = \gamma_0 M q_0.$$
The conjugate-gradient iteration may break down if
\[ \langle p_j, H p_j \rangle = 0, \]
which can occur only if \( H \) is not positive definite, and will stop if \( \langle g_j, v_j \rangle = 0 \). On the other hand, the Lanczos iteration can fail only if \( K_j \) is an invariant subspace for \( M^{-1} H \).

If \( q(s) \) is convex in the manifold \( K_{j+1} \), the minimizer \( s_{j+1} \) of \( q \) in this manifold satisfies
\[
s_{j+1} = s_j + \alpha_j p_j
\]
so long as the initial value \( s_0 = 0 \) is chosen. Thus this estimate easily recurs from the conjugate-gradient iteration. The minimizers in successive manifolds may also be easily obtained using the Lanczos process, although the conjugate-gradient iteration is slightly less expensive and thus preferred.

The vector \( g_{j+1} \) in the conjugate-gradient method gives the gradient of \( q(s) \) at \( s_{j+1} \). It is quite common to stop the method as soon as this gradient is sufficiently small, and the method naturally records the \( M^{-1} \)-norm of the gradient,
\[
\| g_{k+1} \|_{M^{-1}} = \langle g_{k+1}, v_{k+1} \rangle.
\]
(4.22)

There are a number of crucial observations to be made here. First, it is important to note that the resulting trust-region problem involves the two-norm rather than the \( M \)-norm. Second, as \( T_k \) is tridiagonal, it is feasible to use the Moré–Sorensen algorithm to compute the model minimizer even when \( n \) is large. Third, having found \( h_k \), the matrix \( Q_k \) is needed to recover \( s_k \), and thus the Lanczos vectors will either need to be saved on backing store or regenerated. As we shall see, we need only \( Q_k \) once we are satisfied that continuing the Lanczos process will give little extra benefit. Fourth, one would hope that as a sequence of such problems may be solved, and as \( T_k \)
only changes by the addition of an extra diagonal and superdiagonal entry, solution data from one subproblem may be useful for starting the next. We consider this issue in section 5.2.

The basic trust-region solution classification theorem, Theorem 2.1, shows that

\[ (T_k + \lambda_k I_{k+1})h_k = -\gamma_0 e_1, \]

where \(T_k + \lambda_k I_{k+1}\) is positive semidefinite, \(\lambda_k \geq 0\), and \(\lambda_k(\|h_k\|_2 - \Delta) = 0\). What does this tell us about \(s_k\)? First, using (4.17), (4.18), and (5.4) we have

\[ Q_k^T(H + \lambda_k M)s_k = Q_k^T(H + \lambda_k M)Q_k h_k = (T_k + \lambda_k I_{k+1})h_k = -\gamma_0 e_1 = -Q_k^T q \]

and additionally that

\[ \lambda_k(\|s_k\|_M - \Delta) = 0 \quad \text{and} \quad \lambda_k \geq 0. \]

Comparing these with the trust-region classification theorem, we see that \(s_k\) is the Galerkin approximation to \(s^*\) from the space spanned by \(Q_k\).

We may then ask how good the approximation is. In particular, what is the error \((H + \lambda_k M)s_k + g\)? The simplest way of measuring this error would be to calculate \(h_k\) and \(\lambda_k\) by solving (5.3), then to recover \(s_k\) as \(Q_k h_k\), and finally to substitute \(s_k\) and \(\lambda_k\) into \((H + \lambda M)s + g\). However, this is inconvenient as it requires that we have easy access to \(Q_k\). Fortunately there is a far better way.

**Theorem 5.1.**

\[ (H + \lambda_k M)s_k + g = \gamma_{k+1}(e_{k+1}, h_k)w_{k+1} \]

and

\[ \|Q_k^T(H + \lambda M)s_k + g\|_{M^{-1}} = \gamma_{k+1}\|\langle e_{k+1}, h_k\rangle\|. \]

**Proof.** We have that

\[
H s_k = H Q_k h_k \\
= M Q_k T_k h_k + \gamma_{k+1}(e_{k+1}, h_k)w_{k+1} \quad \text{from (4.14)} \\
= -M Q_k (\lambda_k h_k + \gamma_0 e_1) + \gamma_{k+1}(e_{k+1}, h_k)w_{k+1} \quad \text{from (5.4)} \\
= -\lambda_k M Q_k h_k - \gamma_0 M Q_k e_1 + \gamma_{k+1}(e_{k+1}, h_k)w_{k+1} \\
= -\lambda_k M s_k - \gamma_0 M q_0 + \gamma_{k+1}(e_{k+1}, h_k)w_{k+1} \\
= -\lambda_k M s_k - g + \gamma_{k+1}(e_{k+1}, h_k)w_{k+1} \quad \text{from (4.19).}
\]

This then directly gives (5.6). Equation (5.7) follows from the \(M^{-1}\)-orthonormality of \(w_{k+1}\). □

Therefore we can indirectly measure the error (in the \(M^{-1}\)-norm) knowing simply \(\gamma_{k+1}\) and the last component of \(h_k\), and we do not need \(s_k\) or \(Q_k\) at all. Observant readers will notice the strong similarity between this error estimate and the estimate (4.22) for the gradient of the model in the Lanczos method, but this is not at all surprising as the two methods are aiming for the same point if the trust-region radius is large enough. An interpretation of (5.7) is also identical to that of (4.22). The error will be small when either \(\gamma_{k+1}\) or the last component of \(h_k\) is small.

We now consider the problem (5.3) in more detail. We say that a symmetric tridiagonal matrix is **reducible** if one or more of its off-diagonal entries is zero; otherwise it is **irreducible**. We then have the following preliminary result.
Lemma 5.2 (see also [11, Theorem 7.9.5]). Suppose that the tridiagonal matrix \( T \) is irreducible and that \( v \) is an eigenvector of \( T \). Then the first component of \( v \) is nonzero.

Proof. By definition

\[ Tv = \theta v \]  

(5.8)

for some eigenvalue \( \theta \). Suppose that the first component of \( v \) is zero. Considering the first component of (5.8), we have that the second component of \( v \) is zero as \( T \) is tridiagonal and irreducible. Repeating this argument for the \( i \)th component of (5.8), we deduce that the \((i + 1)\)st component of \( v \) is zero for all \( i \) and hence that \( v = 0 \). But this contradicts the assumption that \( v \) is an eigenvector, and so the first component of \( v \) cannot be zero.

This immediately yields the following useful result.

Theorem 5.3. Suppose that \( T_k \) is irreducible. Then the hard case cannot occur for the subproblem (5.3).

Proof. Suppose the hard case occurs. Then, by definition, \( \gamma_0 e_1 \) is orthogonal to \( v_k \), the eigenvector corresponding to the leftmost eigenvalue, \( -\theta_k \), of \( T_k \). Thus, the first component of \( v_k \) is zero, which, following Lemma 5.2, contradicts the assumption that \( v_k \) is an eigenvector. Thus the hard case cannot occur.

This result is important as it suggests that the full power of the Moré–Sorensen [9] algorithm is not needed to solve (5.3). We shall return to this in section 5.2. We also have an immediate corollary.

Corollary 5.4. Suppose that \( T_{n-1} \) is irreducible. Then the hard case cannot occur for the original problem (2.1).

Proof. When \( k = n - 1 \), the columns of \( Q_{n-1} \) form a basis for \( \mathbb{R}^n \). Thus the problems (2.1) and (5.2) are identical and (5.2) and (5.3) are related through a non-singular transformation. The result then follows directly from Theorem 5.3 in the case \( k = n - 1 \).

Thus, if the hard case occurs for (2.1), the Lanczos tridiagonal must become reducible at some stage.

Theorem 5.5. Suppose that \( T_k \) is irreducible, that \( h_k \) and \( \lambda_k \) satisfy (5.4), and that \( T_k + \lambda_k I_{k+1} \) is positive semidefinite. Then \( T_k + \lambda_k I_{k+1} \) is positive definite.

Proof. Suppose that \( T_k + \lambda_k I_{k+1} \) is singular. Then there is a nonzero eigenvector \( v_k \) for which \((T_k + \lambda_k I_{k+1})v_k = 0\). Hence, combining this with (5.4) reveals that

\[ 0 = \langle h_k, (T_k + \lambda_k I_{k+1})v_k \rangle = \langle v_k, (T_k + \lambda_k I_{k+1})h_k \rangle = -\gamma_0 \langle v_k, e_1 \rangle \]

and hence that the first component of \( v_k \) is zero. But this contradicts Lemma 5.2. Hence \( T_k + \lambda_k I_{k+1} \) is both positive semidefinite and nonsingular and thus positive definite.

This result implies that (5.4) has a unique solution. We now consider this solution.

Theorem 5.6. Suppose that \( \langle e_{k+1}, h_k \rangle = 0 \). Then \( T_k \) is reducible.

Proof. Suppose that \( T_k \) is irreducible. As the \((k + 1)\)st component of \( h_k \) is zero, then from the irreducibility of \( T_k \) and the \((k + 1)\)st equation of (5.4), we deduce that the \( k \)th component of \( h_k \) is zero. Repeating this argument for the \((i + 1)\)st equation of (5.4), we deduce that the \( i \)th component of \( h_k \) is zero for \( 1 \leq i \leq k \) and hence that \( h_k = 0 \). But this contradicts the first equation of (5.4), and thus \( T_k \) must be reducible.
Thus we see that of the two possibilities suggested by Theorem 5.1 for obtaining an $s_k$ for which $(H + \lambda_k M) s_k + g = 0$, it will be the possibility $\gamma_{k+1} = 0$ that occurs before $\langle e_{k+1}, h_k \rangle = 0$.

**Theorem 5.7.** Suppose that the hard case does not occur for (2.1), and that $\gamma_{k+1} = 0$. Then $s_k$ solves (2.1).

**Proof.** If $\gamma_{k+1} = 0$, the Krylov space $K_k$ is an invariant subspace of $M^{-1}H$, and by construction the first basis element of this space is $M^{-1}g$. As the hard case does not occur for (2.1), this space must also contain at least one eigenvector corresponding to the leftmost eigenvalue, $-\theta$, of $M^{-1}H$. Thus one of the eigenvalues of $T_k$ must be $-\theta$, and $\lambda_k \geq \theta$ as $T_k + \lambda_k I_{k+1}$ is positive semidefinite. But this implies that $H + \lambda_k M$ is positive semidefinite, which combines with (5.1), (5.5), and Theorem 5.1 with $\gamma_{k+1} = 0$ to show that $s_k$ satisfies the optimality conditions shown in Theorem 2.1. □

Thus we see that in the easy case, the required solution will be obtained from the first irreducible block of the Lanczos tridiagonal. It remains for us to consider the hard case. In view of Corollary 5.4, this case can only occur when $T_k$ is reducible. Suppose therefore that $T_k$ reduces into $\ell$ blocks of the form

$$(5.9) \quad T_k = \begin{pmatrix} T_{k_1} & \cdots & \cdots \\ & T_{k_2} & \\ & & \ddots \\ & & & T_{k_\ell} \end{pmatrix},$$

where each of the $T_{k_i}$ defines an invariant subspace for $M^{-1}H$ and where the last block $T_{k_\ell}$ is the first to yield the leftmost eigenvalue, $-\theta$, of $M^{-1}H$. Then there are two cases to consider.

**Theorem 5.8.** Suppose that the hard case occurs for (2.1), that $T_k$ is as described by (5.9), and that the last block $T_{k_\ell}$ is the first to yield the leftmost eigenvalue, $-\theta$, of $M^{-1}H$. Then,

1. if $\theta \leq \lambda_{k_1}$, a solution to (2.1) is given by $s_k = Q_k h_{k_1}$, where $h_{k_1}$ solves the positive-definite system

$$ (T_{k_1} + \lambda_{k_1} I_{k_1+1}) h_{k_1} = -\gamma_0 e_1; $$

2. if $\theta > \lambda_{k_1}$, a solution to (2.1) is given by $s_k = Q_k h_k$, where

$$ (5.10) \quad h_k = \begin{pmatrix} h \\ 0 \\ \vdots \\ 0 \\ \alpha u \end{pmatrix}, $$

$h$ is the solution of the nonsingular tridiagonal system

$$ (T_{k_1} + \theta I_{k_1+1}) h = -\gamma_0 e_1, $$

$u$ is an eigenvector of $T_{k_\ell}$ corresponding to $-\theta$, and $\alpha$ is chosen so that

$$ \|h_{k_1}\|^2 + \alpha^2 \|u\|^2 = \Delta^2. $$

**Proof.** In case 1, $H + \lambda_{k_1} M$ is positive semidefinite as $\lambda_{k_1} \geq \theta$ and the remaining optimality conditions are satisfied as $\gamma_{k_1+1} = 0$ and $h_{k_1}$ solves (5.2). That $T_{k_1} + \lambda_{k_1} I_{k_1+1}$ is positive definite follows from Theorem 5.5. In case 2, $H + \theta M$ is positive.
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Furthermore, as θ > λ_k, it is easy to show that ∥h∥2 < ∥h_k∥2 ≤ Δ and hence that there is a root α for which ∥s_k∥_M = ∥h_k∥2 = Δ. Finally, as each Q_k defines an invariant subspace, HQ_k = MQ_kT_k. Writing s = Q_kh and v = Q_ku, we therefore have

Hs = HQ_kh = MQ_kT_kh = MQ_k(−θh − γ_0e_1) = −θMs − g

and

Hv = HQ_ku = MQ_kT_ku = −θMQ_ku = −θMv.

Thus (H + θM)s_k = −g and s_k satisfies all optimality conditions for (5.2).

Notice that to obtain s_k as described in this theorem, we require only the Lanczos vectors corresponding to blocks one and, perhaps, ℓ of T_k.

We do not claim that solving the problem as outlined in Theorem 5.8 is realistic, as it relies on our being sure that we have located the leftmost eigenvalue of M⁻¹H. With Lanczos-type methods, one cannot normally guarantee that all eigenvalues, including the leftmost, will be found unless one ensures that all invariant subspaces have been investigated, and this may prove to be very expensive for large problems. In particular, the Lanczos algorithm, Algorithm 4.2, terminates each time an invariant subspace has been determined and must be restarted using a vector q which is M-orthonormal to the previous Lanczos directions. Such a vector may be obtained from the Gram–Schmidt process by reorthonormalizing a suitable vector—a vector with some component M-orthogonal to the existing invariant subspaces, perhaps a random vector—with respect to the previous Lanczos directions, which means that these directions will have to be regenerated or reread from backing store. Thus, while this form of the solution is of theoretical interest, it is unlikely to be of practical interest if a cheap approximation to the solution is all that is required.

5.1. The algorithm. We may now outline our algorithm, Algorithm 5.1, the generalized Lanczos trust-region (GLTR) method. We stress that, as our goal is merely to improve upon the value delivered by the Steihaug–Toint method, we do not use the full power of Theorem 5.8 and are content just to investigate the first invariant subspace produced by the Lanczos algorithm. In almost all cases, this subspace contains the global solution to the problem, and the complications and costs required to implement a method based on Theorem 5.8 are, we believe, prohibitive in our context.

Algorithm 5.1 (the GLTR method). Let s_0 = 0, g_0 = g, v_0 = M⁻¹g_0, γ_0 = √⟨v_0, g_0⟩, and p_0 = −v_0. Set the flag INTERIOR as true. For k = 0, 1, . . . until convergence, perform the iteration

α_k = ⟨g_k, v_k⟩/⟨p_k, Hp_k⟩

Obtain T_k from T_{k−1} using (4.20)

If INTERIOR is true, but α_k ≤ 0 or ∥s_k + α_kp_k∥_M ≥ Δ

reset INTERIOR to false.

If INTERIOR is true
s_{k+1} = s_k + α_kp_k
else

solve the tridiagonal trust-region subproblem (5.3) to obtain h_k

end if

q_{k+1} = g_k + α_kHp_k
v_{k+1} = M⁻¹q_{k+1}
If \textsc{interior} is true
test for convergence using the residual $\|g_{k+1}\|_{M^{-1}}$

else
test for convergence using the value $\gamma_{k+1}|\langle e_{k+1}, h_k \rangle|$
end if

$\beta_k = \langle g_{k+1}, v_{k+1} \rangle / \langle g_k, v_k \rangle$

$p_{k+1} = -v_{k+1} + \beta_k p_k$

If \textsc{interior} is false, recover $s_k = Q_k h_k$ by rerunning the recurrences or obtaining $Q_k$ from backing store.

When recovering $s_k = Q_k h_k$ by rerunning the recurrences, economies can be made by saving the $\alpha_i$ and $\beta_i$ during the first pass and reusing them during the second. A potentially bigger savings may be made if one is prepared to accept a slightly inferior value of the objective function. The idea is simply to save the value of $q$ at each iteration. On convergence, one looks back through this list to find an iteration, $\ell$, say, for which a required percentage of the best value was obtained, recompute $h_\ell$, and then accept $s_\ell = Q_\ell h_\ell$ as the required estimate of the solution. If the required percentage occurs at an iteration before the boundary is encountered, both the final point before the boundary and the Steihaug–Toint point are suitable and available without the need for the second pass.

We note that we have used the conjugate-gradient method (Algorithm 4.1) to generate the Lanczos vectors. If the inner-product $\langle p_k, H p_k \rangle$ proves to be tiny, it is easy to continue using the Lanczos method (Algorithm 4.2) itself; the vectors $q_j = v_j / \sqrt{\langle g_j, v_j \rangle}$ and $w_j = g_j / \sqrt{\langle g_j, v_j \rangle}$ required to continue the Lanczos recurrence (4.11) are directly calculable from the conjugate-gradient method.

At each stage of both the Steihaug–Toint algorithm and our GLTR method (Algorithm 5.1), we need to calculate $\|s_k + \alpha p_k\|_M$. This issue is not discussed by Steihaug as it is implicitly assumed that $M$ is available. However, it may be the case that all that is actually available is a procedure which returns $M^{-1}v$ for a given input $v$, and thus $M$ is unavailable. Fortunately this is not a significant drawback as it is possible to calculate $\|s_k + \alpha p_k\|_M$ from available information.

To see this, observe that

\begin{equation}
\|s_k + \alpha p_k\|_M^2 = \|s_k\|_M^2 + 2\alpha \langle s_k, M p_k \rangle + \alpha^2 \|p_k\|_M^2 \tag{5.11}
\end{equation}

and thus that we can find $\|s_{k+1}\|_M^2$ from $\|s_k\|_M^2$ so long as we already know $\langle s_k, M p_k \rangle$ and $\|p_k\|_M^2$. But it is straightforward to show that these quantities may be calculated from the pair of recurrences

\begin{align}
\langle s_k, M p_k \rangle &= \beta_{k-1} \left( \langle s_{k-1}, M p_{k-1} \rangle + \alpha_{k-1} \|p_{k-1}\|_M^2 \right) \tag{5.12} \\
\|p_k\|_M^2 &= \langle g_k, v_k \rangle + \beta_k^2 \|p_{k-1}\|_M^2. \tag{5.13}
\end{align}

where, of course, $\langle g_k, v_k \rangle$ has already been calculated as part of the preconditioned conjugate-gradient method.

5.2. Solving the irreducible tridiagonal trust-region subproblem. In view of Theorem 5.3, the irreducible tridiagonal trust-region subproblem (5.3) is, in theory, easier to solve than the general problem. This is so both because the Hessian is tridiagonal (and thus very inexpensive to factorize) and because the hard case
cannot occur. We should be cautious here because the so-called almost hard case—which occurs when \( g \) only has a tiny component in the range-space of \( H(\lambda^m) \)—may still happen, and the trust-region problem in this case is naturally ill conditioned and thus likely to be difficult to solve.

The Moré–Sorensen [9] algorithm is based on being able to form factorizations of the model Hessian (which is certainly the case here as \( T_k + \lambda I_{k+1} \) is tridiagonal), but does not try to calculate the leftmost eigenvalue of the pencil \( H + \lambda M \). In the tridiagonal case, computing the extreme eigenvalues is straightforward, particularly if a sequence of related problems is to be solved. Thus, rather than using the Moré–Sorensen algorithm, we prefer the following method.

We restrict ourselves to the case where the solution lies on the trust-region boundary—we will only switch to this approach when the conjugate-gradient iteration leaves the trust region. The basic iteration is identical to that proposed in [9], namely, to apply Newton’s method to

\[
\phi(\lambda) \overset{\text{def}}{=} \frac{1}{\|h_k(\lambda)\|_2} - \frac{1}{\Delta} = 0,
\]

where

\[
(T_k + \lambda I_{k+1})h_k(\lambda) = -\gamma_0 e_1,
\]

to find the required root \( \lambda_k \). Recalling that we denote the leftmost eigenvalue of \( T_k \) by \( -\theta_k \), the main difference between our approach and Moré and Sorensen’s is that we always start from some point in the interval \( \max(0, \theta_k, \lambda_k) \)—this interval is characterized by both \( T_k + \lambda I_{k+1} \) being positive definite and \( \|h_k(\lambda)\|_2 \geq \Delta \)—as then the resulting Newton iteration is globally linearly, and asymptotically quadratically, convergent without any further safeguards. The Newton iteration is performed using Algorithm 5.2.

**Algorithm 5.2 (Newton’s method to solve \( \phi(\lambda) = 0 \)).** Let \( \lambda > \theta_k \) and \( \Delta > 0 \) be given.

1. Factorize \( T_k + \lambda I_{k+1} = BDB^T \), where \( B \) and \( D \) are unit bidiagonal and diagonal matrices, respectively.
2. Solve \( BDB^T h = -\gamma_0 e_1 \).
3. Solve \( Bw = h \).
4. Replace \( \lambda \) by

\[
\lambda + \left( \frac{\|h\|_2 - \Delta}{\Delta} \right) \left( \frac{\|h\|_2^2 - \Delta}{\|w\|_{D^{-1}}^2} \right).
\]

The Newton correction in step 4 of this algorithm is given by

\[
\lambda - \phi(\lambda)\phi'(\lambda) = \lambda + \left( \frac{\|h\|_2 - \Delta}{\Delta} \right) \left( \frac{\|h\|_2^2 - \Delta}{\langle h, (T_k + \lambda I_{k+1})^{-1} h \rangle} \right),
\]

while the exact form given is obtained by using the identity

\[
\langle h, (T_k + \lambda I_{k+1})^{-1} h \rangle = \langle h, B^{-T}D^{-1}B^{-1}h \rangle = \langle B^{-1}h, D^{-1}B^{-1}h \rangle = \|w\|_{D^{-1}}^2,
\]

where \( w \) is as computed in step 3. It is slightly more efficient to pick \( B \) to be unit upper-bidiagonal rather than unit lower-bidiagonal, as then step 2 simplifies to \( B^T h = -\gamma_0 D^{-1}e_1 \) because of the structure of the right-hand side.
To obtain a suitable starting value, two possibilities are considered. First, we attempt to use the solution value $\lambda_{k-1}$ from the previous subproblem. Recall that $T_k$ is merely $T_{k-1}$ with an appended row and column. As we already have a factorization of $T_{k-1} + \lambda_{k-1}I_k$, it is trivial to obtain that of $T_k + \lambda_{k-1}I_{k+1}$ and thus to determine if the latter is positive definite. If $T_k + \lambda_{k-1}I_{k+1}$ turns out to be positive definite, $h_k(\lambda_{k-1})$ is computed from (5.15), and if $\|h_k(\lambda_{k-1})\|_2 \geq \Delta$, $\lambda_{k-1}$ is used to start the Newton iteration.

Second, if $\lambda_{k-1}$ is unsuitable, we monitor $T_k$ to see if it is indefinite. This is trivial, as, for instance, the matrix is positive definite so long as all of the $\alpha_i$, $0 \leq i \leq k$, generated by the conjugate-gradient method are positive. If $T_k$ is positive definite, we start the Newton iteration with the value $\lambda = 0$, which by assumption gives $\|h_k(0)\|_2 \geq \Delta$ as the unconstrained solution lying outside the trust region. Otherwise, we determine the leftmost eigenvalue, $-\theta_k$, of $T_k$ and start with $\lambda = \theta + \epsilon$, where $\epsilon$ is a small positive number chosen to make $T_k + \lambda_{k-1}I_{k+1}$ numerically “just” positive definite. By this we mean that its $BDB^T$ factorization should exist, but that $\epsilon$ should be as small as possible. We have found that a value $(1 + \theta_k)\epsilon_m$, where $\epsilon_m$ is the unit roundoff, is almost always suitable, but we have added the precaution of multiplying this value by increasing powers of 2 so long as the factorization fails.

If we need to compute the leftmost eigenvalue of $T_k$, we use an iteration based upon the last-pivot function proposed by Parlett and Reid [12]. The last-pivot function, $\delta_k(\theta)$, is simply the value of the last diagonal entry of the $BDB^T$ factor $D_k(\lambda)$ of $T_k - \theta I_{k+1}$. This value will be zero, and the other diagonal entries positive, when $\theta = \theta_k$ and $\delta_k(\theta) > 0$ for $\theta \geq \theta_k$. An interval of uncertainty $[\theta_i, \theta_u]$ is placed around the required root. The initial interval is given by the Gersgorin bounds on the leftmost eigenvalue. When it is known, the leftmost eigenvalue, $-\theta_{k-1}$, of $T_{k-1}$ may be used to improve the lower bound because of the Cauchy interlacing property of the eigenvalues of $T_{k-1}$ and $T_k$ (see, for instance, [11, Theorem 10.1.2]). Given an initial estimate of $\theta_k$, an improvement may be sought by applying Newton’s method to $\delta_k(\theta)$; the derivative of $\delta_k$ is easy to obtain by recurrence. However, as Parlett and Reid point out,

$$\delta_k(\theta) = \frac{\det(T_k - \theta I_{k+1})}{\det(T_{k-1} - \theta I_k)}$$

and thus has a pole at $\theta = \theta_{k-1}$. Hence it is better to choose the new point by fitting the model

$$\delta_k^a(\theta) = \frac{(\theta - a)(\theta - b)}{\theta - \theta_{k-1}}$$

(5.16)

to the function and derivative value at the current $\theta$ and then to pick the new iterate as the larger root of $\delta_k^a(\theta)$. If the new iterate lies outside the interval of uncertainty, it is replaced by the midpoint of the interval. The interval is then contracted by computing $\delta_k$ at the new iterate and replacing the appropriate endpoint by the iterate. The iteration is stopped if the length of the interval or the value of $\delta_k(\theta_k)$ is small.

If $\theta_{k-1}$ is known, the initial iterate chosen as $\theta_{k-1} + \epsilon$ for some small positive $\epsilon \leq \theta_k - \theta_{k-1}$, and successive iterates generated from (5.16), the iterates converge globally, and asymptotically superlinearly, from the left. If the Newton iteration is used, the required root is frequently obscured and the scheme resorts to interval bisection. Thus the Parlett–Reid scheme is preferred.
Other means of locating the required eigenvalue based on using the determinant \( \det(T_k - \theta I_k + 1) \) instead of \( \delta_k(\theta) \) were tried, but proved to be less reliable because of the huge numerical range (and thus potential overflow) of the determinant.

6. Numerical experiments. The algorithm sketched in sections 5.1 and 5.2 has been implemented as a Fortran-90 module, \texttt{HSL VF05}, within the Harwell Subroutine Library [6].

As our main interest is in using the methods described in this paper within a trust-region algorithm, we are particularly concerned with two issues. First, can we obtain significantly better values of the model by finding better approximations to its solution than the Steihaug–Toint method? And second, do better approximations to the minimizer of the model necessarily translate into fewer iterations of the trust-region method? In this section, we address these outstanding questions.

Throughout, we will consider the basic problem of minimizing an objective \( f(x) \) of \( n \) real variables \( x \). We shall use the following standard trust-region method.

\textbf{Algorithm 6.1 (standard trust-region algorithm).}

0. An initial point \( x_0 \) and an initial trust-region radius \( \Delta_0 \) are given, as are constants \( \epsilon_g, \eta_1, \eta_2, \gamma_1, \) and \( \gamma_2 \), which are required to satisfy the conditions

\begin{equation}
0 < \eta_1 \leq \eta_2 < 1 \text{ and } 0 < \gamma_1 < 1 \leq \gamma_2.
\end{equation}

Set \( k = 0 \).

1. Stop if \( \| \nabla f (x_k) \|_2 \leq \epsilon_g \).

2. Define a second-order Taylor series model \( q_k \) and a positive-definite preconditioner \( M_k \). Compute a step \( s_k \) to “sufficiently reduce the model” \( q_k \) within the trust region \( \| s \|_{M_k} \leq \Delta_k \).

3. Compute the ratio

\begin{equation}
\rho_k = \frac{f(x_k) - f(x_k + s_k)}{q_k(x_k) - q_k(x_k + s_k)}.
\end{equation}

If \( \rho_k \geq \eta_1 \), let \( x_{k+1} = x_k + s_k \); otherwise let \( x_{k+1} = x_k \).

4. Set

\begin{equation}
\Delta_{k+1} = \begin{cases} 
\gamma_2 \Delta_k & \text{if } \rho_k \geq \eta_2, \\
\Delta_k & \text{if } \rho_k \in [\eta_1, \eta_2), \\
\gamma_1 \Delta_k & \text{if } \rho_k < \eta_1.
\end{cases}
\end{equation}

Increment \( k \) by one, and go to step 1.

We choose the specific values \( \epsilon_g = 0.00001, \eta_1 = 0.01, \eta_2 = 0.95, \gamma_1 = 0.5, \) and \( \gamma_2 = 2 \) and set an upper limit of \( n \) iterations. The step \( s_k \) in step 2 is computed using either Algorithm 5.1 or the Steihaug–Toint algorithm. Convergence in both algorithms for the subproblem occurs as soon as

\begin{equation}
\| g_{k+1} \|_{M^{-1}} \leq \min(0.1, \| g_0 \|^0_{M^{-1}}) \| g_0 \|_{M^{-1}}
\end{equation}

or if more than \( n \) iterations have been performed. In addition, of course, the Steihaug–Toint algorithm terminates as soon as the boundary is crossed.

All our tests were performed on an IBM RISC System/6000 3BT workstation with 64 Megabytes of RAM; the codes are all double precision Fortran-90, compiled under xlf90 with -O optimization, and IBM library BLAS are used. The test examples we consider are the larger examples from the \texttt{CUTE} test set [1] for which negative
curvature is frequently encountered. Tests were terminated if more than 30 CPU minutes elapsed.

6.1. Can we get much better model values than Steihaug–Toint? We first consider problems of the form (2.1). Our test examples are generated by running Algorithm 6.1 on the CUTE set for 10 iterations and taking the trust-region subproblem at iteration 10 as our example. The idea here is to simulate the kind of subproblems which occur in practice, not those which result at the starting point for the algorithm, as such points frequently have special (favorable) properties.

Our aim is to see whether there is any significant advantage in continuing the minimization of the trust-region subproblem once the boundary of the trust region has been encountered. We ran HSL_VF05 to convergence, stopping when \( \|g_{k+1}\|_{M^{-1}} \leq \max(10^{-15}, 10^{-5}\|g_0\|_{M^{-1}}) \) or more than \( n \) iterations had been performed.

In all of the experiments reported here, the best value found was in fact the optimum value—a factorization of \( H + \lambda M \) was used to confirm that the matrix was positive semidefinite, while the algorithm ensured that the remaining optimality conditions held—although, of course, there is no guarantee that this will always be the case. We measured the iteration (ST) and the percentage (ratio) of the optimal value obtained at the point at which the Steihaug–Toint method left the trust region, as well as the number of iterations taken to achieve 10%, 90%, and 99% of the optimal reduction (10%, 90%, 99%, respectively).

The results of these experiments are summarized in Table 6.1. In this table we give the name of each example used, along with its dimension \( n \), and the statistics “ratio” (expressed in the form \( x(y) \) as a shorthand for \( x \times 10^y \)), “ST,” “10%,” “90%,” and “99%” as just described. Some of the problems had interior solutions, in which case the “ratio” and “ST” statistics are absent (as indicated by a dash). We considered both the unpreconditioned method (\( M = I_n \)) and a variety of standard preconditioners—a band preconditioner with semibandwidth of 5, and modified incomplete and sparse Cholesky factorizations, with the modifications as proposed in [18]—used by the LANCELOT package (see [2, Chapter 3]). The Cholesky factorization methods both failed for the problem MSQRTALS for which the Hessian matrix required too much storage.

We make a number of observations.
1. On some problems, the Steihaug–Toint point gives a model value which is a good approximation to the optimal value.
2. On other problems, a few extra iterations beyond the Steihaug–Toint point pay handsome dividends.
3. Getting to within 90% or even 99% of the best value very rarely requires many more iterations than to find the Steihaug–Toint point.

In conclusion, based on these numbers, we suggest that a good strategy would be to perform a few (say, 5) iterations beyond the Steihaug–Toint point and accept the improved point only if its model value is significantly better (as this will cost a second pass to compute the Lanczos vectors). We shall consider this further in the next section.

6.2. Do better values than Steihaug–Toint imply a better trust-region method? We now consider how the methods we have described for approximately solving the trust-region subproblem perform within a trust-region algorithm. Of particular interest is the question whether solving the subproblem more accurately reduces the number of trust-region iterations or more particularly the cost of solving the problem—the number of iterations is of concern if the evaluation of the objective
A comparison of the number of iterations required to achieve a given percentage of the optimal model value for a variety of preconditioners. See the text for a key to the data.

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<td>1</td>
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<td>8E(-14)</td>
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</tbody>
</table>

function and its derivatives is the dominant cost, as then there is a direct correlation between the number of iterations and the overall cost of solving the problem.

In Tables 6.2 and 6.3, we compare the Steihaug-Toint scheme with the GLTR algorithm (Algorithm 5.1) run to high accuracy. We exclude the problem HYD2C0LS for our reported results, as no method succeeded in solving the problem in fewer than our limit of \( n \) iterations, and the problems BROYDN7D and SPMSRTLS, as a number of different local minima were found. In these tables, in addition to the name and dimension of each example, we give the number of objective function (“#f”) and derivative (“#g”) values computed, the total number of matrix-vector products (“#prod”) required to solve the subproblems, and the total CPU time required in seconds. We compare the same preconditioners \( M \) as we used in the previous section. We indicate those cases where one or another method performs at least 10% better than its competitor by highlighting the relevant figure in bold.
Table 6.2
A comparison of the Steihaug–Toint and exact model minimization techniques within a trust-region method, using a variety of preconditioners, for unconstrained minimization (part 1). See the text for a key to the data.

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<th>#g</th>
<th>#prods</th>
<th>CPU</th>
<th>#f</th>
<th>#g</th>
<th>#prods</th>
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</table>

We observe the following:

1. The use of different $M$ leads to radically different behavior. Different preconditioners appear to be particularly suited to different problems. Surprisingly, perhaps, the unpreconditioned algorithm often performs the best overall.

2. In the unpreconditioned case, the model-optimum variant frequently requires significantly fewer function evaluations than the Steihaug–Toint method. However, the extra algebraic costs per iteration often outweigh the reduction in the numbers of iterations. The advantage in function calls for the other preconditioners is less pronounced.

Ideally, one would like to retain the advantage in numbers of function calls while reducing the cost per iteration. As we noted in section 6.1, one normally gets a good approximation to the optimal model value after a modest number of iterations. Moreover, while the Steihaug–Toint point often gives a significantly suboptimal value, a few extra iterations usually suffice to give a large percentage of the optimum. Thus, we next investigate both of these issues in the context of an overall trust-region method.

In Tables 6.4 and 6.5, we compare the number of function evaluations (#f) and the CPU time taken to solve the problem for the Steihaug–Toint (“ST”) method.
with a number of variations on our basic GLTR method (Algorithm 5.1). The basic requirement is that we compute a model value which is at least 90% of the best value found during the first pass of the GLTR method. If this value is obtained by an iterate before that which gives the Steihaug–Toint point, the Steihaug–Toint point is accepted. Otherwise, a second pass is performed to recover the first point at which 90% of the best value was observed. The other ingredient is the choice of the stopping rule for the first pass. One possibility is to stop this pass as soon as the test (6.4) is satisfied. We denote this strategy by “90%best.” The other possibility is to stop when either (6.4) is satisfied or at most a fixed number of iterations beyond the Steihaug–Toint point have occurred. We refer to this as “90%(ST+k),” where \( k \) gives the number of additional iterations allowed. We investigate the cases \( k = 1, 5, \) and 10. Once again, we compare the same preconditioners \( M \) as we used in the previous section. We highlight in bold those entries which are at least 10% better than the competition.

The conclusions are as broad as before. Each method has its successes and failures, and there is no clear overall best method or preconditioner, although the unpreconditioned version performs surprisingly well. Restricting the number of iterations allowed

### Table 6.3

A comparison of the Steihaug–Toint and exact model minimization techniques within a trust-region method, using a variety of preconditioners, for unconstrained minimization (part 2). See the text for a key to the data.

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<td>&gt; n iterations</td>
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### Table 6.3 (continued)

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<td>&gt; n iterations</td>
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</tr>
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</table>
after the Steihaug–Toint point has been found appears to curb the worst behavior of the unrestricted method.

7. Perspectives and conclusions. We have considered a number of methods which aim to find a better approximation to the solution of the trust-region subproblem than that delivered by the Steihaug–Toint scheme. These methods are based on solving the subproblem within a subspace defined by the Krylov space generated by the conjugate-gradient and Lanczos methods. The Krylov subproblem has a number of useful properties which lead to its efficient solution. The resulting algorithm is available as a Fortran-90 module, HSL_VF05 [6].

We must admit to being slightly disappointed that the new method did not perform uniformly better than the Steihaug–Toint scheme, and we were genuinely surprised that a more accurate approximation does not appear to significantly reduce the number of function evaluations within a standard trust-region method, at least in the tests we performed. While this may limit the use of the methods developed here, it also calls into question a number of other recent eigensolution-based proposals for solving the trust-region subproblem (see [15], [16], [17], [21]). While these authors demonstrate that their methods provide an effective means of solving the subproblem,

### Table 6.4

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5 band ST

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they make no effort to evaluate whether this is actually useful within a trust-region method. The results given in this paper suggest that this may not in fact be the case. This also leads to the interesting question as to whether it is possible to obtain useful low-accuracy solutions with these methods. We believe that further testing is needed to confirm the trends we have observed here.

We should not pretend that the formulae given in this paper are exact or even accurate in floating-point arithmetic. Indeed, it is well known that the floating-point matrices $Q_k$ from the Lanczos method quickly lose $M$-orthonormality (see, for instance, [11, Section 13.3]). Despite this, the method as given appears to be capable of producing usable approximate solutions to the trust-region subproblem. We are currently investigating why this should be so.

One further possibility, which we have not considered so far, is to find an estimate $\lambda$ using the first pass of Algorithm 5.1 and then to compute the required $s$ by minimizing the unconstrained model $\langle g, s \rangle + \frac{1}{2} \langle s, (H + \lambda M)s \rangle$ using the preconditioned conjugate-gradient method. The advantage of doing this is that any instability in the first pass does not necessarily reappear in this auxiliary calculation. The disadvantages are that it may require more work than simply using (5.1) and that $\lambda$ must be
computed sufficiently large to ensure that $H + \lambda M$ is positive semidefinite.

**Acknowledgments.** We would like to thank John Reid for his helpful advice on computing eigenvalues of tridiagonal matrices and Jorge Moré for his useful comments on the Moré–Sorensen [9] method. Many thanks are also due to Jorge Nocedal and two anonymous referees.

**REFERENCES**

