Notes for Part 4: Interior-point methods for inequality constrained optimization

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4 Sketches of proofs for Part 4

4.1 Proof of Theorem 4.1

Let $\mathcal{A} = \mathcal{A}(x_*)$, and $\mathcal{I} = \{1, \ldots, m\} \setminus \mathcal{A}$ be the indices of constraints that are active and inactive at x_*. Furthermore let subscripts $_\mathcal{A}$ and $_\mathcal{I}$ denote the rows of matrices/vectors whose indices are indexed by these sets. Denote the left generalized inverse of $A^T_\mathcal{A}(x)$ by

$$A^+_\mathcal{A}(x) = \left(A_\mathcal{A}(x) A^T_\mathcal{A}(x) \right)^{-1} A_\mathcal{A}(x)$$

at any point for which $A_\mathcal{A}(x)$ is full rank. Since, by assumption, $A_\mathcal{A}(x_*)$ is full rank, these generalized inverses exists, and are bounded and continuous in some open neighbourhood of x_*.

Now let

$$(y_k)_i = \frac{\mu_k}{c_i(x_k)}$$

for $i = 1, \ldots, m$, as well as

$$(y_*)_\mathcal{A} = A^+_\mathcal{A}(x_*) g(x_*)$$

and $(y_*)_\mathcal{I} = 0$. If $\mathcal{I} \neq \emptyset$,

$$\|(y_k)_\mathcal{I}\|_2 \leq 2\mu_k \sqrt{|\mathcal{I}|} / \min_{i \in \mathcal{I}} |c_i(x_*)| \tag{4.1}$$

for all sufficiently large k. It then follows from the inner-iteration termination test that

$$\begin{aligned} \|g(x_k) - A^T_\mathcal{A}(x_k)(y_k)_\mathcal{A}\|_2 \leq & \; \|g(x_k) - A^T(x_k) y_k\|_2 + \|A^T_\mathcal{I}(x_k)(y_k)_\mathcal{I}\|_2 \\ \leq & \; \bar{\epsilon}_k \stackrel{\text{def}}{=} \epsilon_k + \mu_k \frac{2\sqrt{|\mathcal{I}|}\|A_\mathcal{I}\|_2}{\min_{i \in \mathcal{I}} |c_i(x_*)|}. \end{aligned} \tag{4.2}$$

Hence

$$\|A^+_\mathcal{A}(x_k) g(x_k) - (y_k)_\mathcal{A}\|_2 = \|A^+_\mathcal{A}(x_k)(g(x_k) - A^T_\mathcal{A}(x_k)(y_k)_\mathcal{A})\|_2 \leq 2\|A^+_\mathcal{A}(x_*)\|_2 \bar{\epsilon}_k.$$

Then

$$\|(y_k)_\mathcal{A} - (y_*)_\mathcal{A}\|_2 \leq \|A^+_\mathcal{A}(x_*) g(x_*) - A^+_\mathcal{A}(x_k) g(x_k)\|_2 + \|A^+_\mathcal{A}(x_k) g(x_k) - (y_k)_\mathcal{A}\|_2$$

which, in combination with (4.1), implies that $\{y_k\}$ converges to y_*. In addition, continuity of the gradients and (4.2) implies that

$$g(x_*) - A^T(x_*) y_* = 0$$

while the fact that $c(x_k) > 0$ for all k, the definition of y_k and y_* (and the implication that $c_i(x_k)(y_k)_i = \mu_k$) shows that $c(x_*) \geq 0$, $y_* \geq 0$ and $c_i(x_*)(y_*)_i = 0$. Hence (x_*, y_*) satisfies the first-order optimality conditions.

4.2 Proof of Theorem 4.2

The proof of this result is elementary, but rather long and involved. See

> N. Gould, D. Orban, A. Sartenaer and Ph. L. Toint, "Superlinear convergence of primal-dual interior point algorithms for nonlinear programming", *SIAM J. Optimization*, **11**(4) (2001) 974:1002

for full details.