Bi-Parametric Operator Preconditioning: Theory and Applications

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joint work with

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Outline

1. Motivation
2. Abstract Setting
3. Bi-Parametric Operator Preconditioning
4. Iterative Solvers Performance: Hilbert space setting
   - Linear convergence
   - Super-linear convergence

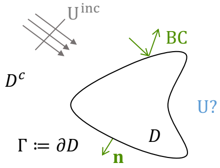
Motivation: The Electric Field Integral Equation

Consider $D^c \in \mathbb{R}^3$ unbounded Lipschitz with boundary Γ, wavenumber $\kappa > 0$

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(\mathbf{x}) - \kappa^2 \mathbf{U}(\mathbf{x}) = \mathbf{0} \qquad \mathbf{x} \in D^c$$

$$\mathbf{n} \times \mathbf{U} = -\mathbf{n} \times \mathbf{U}^{\text{inc}} \qquad \mathbf{x} \in \Gamma \quad \text{(PEC)}$$

$$|\mathbf{curl}\,\mathbf{U}(\mathbf{x}) \times \hat{\mathbf{x}} - \imath\kappa \mathbf{U}(\mathbf{x})| = o(\|\mathbf{x}\|^{-1}) \qquad \text{(Silver-Müller)}$$



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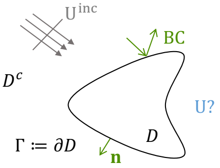

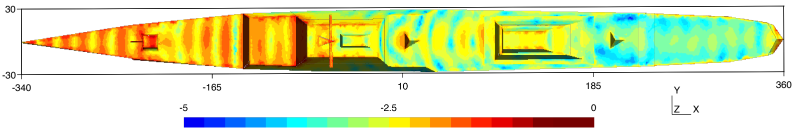

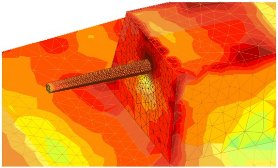

J-H, Escapil, IEEE TAP 2019

Motivation: The Electric Field Integral Equation (EFIE)

Reduce the BVP to its boundary: we use the EFIE for the surface current $\mathbf{j}$ is

$$\mathcal{T}(\mathbf{j}) := \imath\omega \int_{\Gamma} \frac{e^{-\imath\kappa\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|}\mathbf{j}(\mathbf{y})d\mathbf{y} - \frac{1}{\imath\omega\epsilon}\mathbf{n}\times\nabla\int_{\Gamma}\frac{e^{-\imath\kappa\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|}\nabla_{\Gamma}\cdot\mathbf{j}(\mathbf{y})d\mathbf{y} = -\mathbf{n}\times\mathbf{U}^{\text{inc}}$$

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Preconditioning Needed!

Motivation: Solving the EFIE

- Idea: Operator preconditioner in the form of **Calderón** preconditioning

$$(\mathcal{T} \circ \mathcal{T})(\mathbf{j}) = \mathcal{T}^2(\mathbf{j}) = -\frac{1}{4}\mathbf{j} + \mathcal{K}^2(\mathbf{j})$$

where

$$\mathcal{K}(\mathbf{j}) := \mathbf{n} \times \nabla_{\mathbf{x}} \times \int_\Gamma \frac{e^{-\imath\kappa\|\mathbf{x}-\mathbf{y}\|}}{4\pi\|\mathbf{x}-\mathbf{y}\|}\mathbf{j}(\mathbf{y})d\mathbf{y} \quad (\text{"compact"})$$

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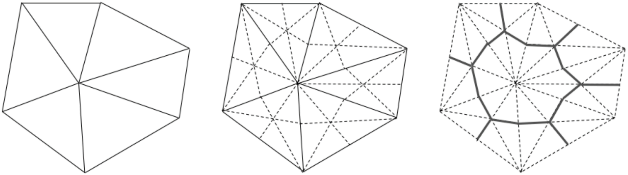
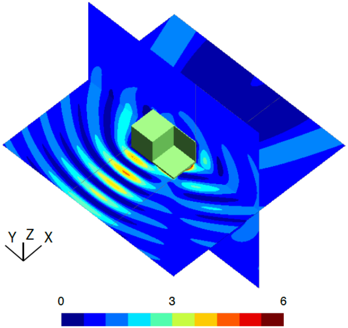
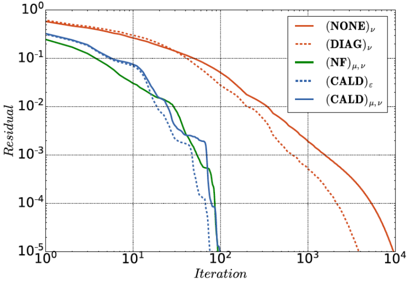
- Dual mesh achieved via barycentric refinement leads to six-fold increase in computational cost



Primal Mesh — Barycentric Mesh — Dual Mesh

EM Fichera cube solved with GMRES(200) with $k = 10$, $r = 10$, $N = 16'113$ and $N_b = 96'678$





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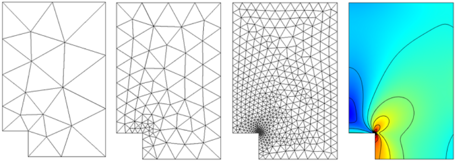
YES WE CAN

★ ★ ★ ★ ★

MAKE OPERATOR PRECONDITIONING GREAT AGAIN

Outline

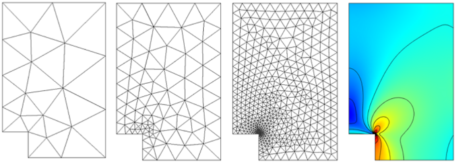
Petrov-Galerkin Methods



1. **Continuous operator analysis:**
   X, Y reflexive Banach spaces, $\mathsf{a} \in \mathcal{L}(X \times Y; \mathbb{C})$, $b \in Y'$.

$$\text{Seek} \quad u \in X \quad \text{such that} \quad \mathsf{a}(u, v) = b(v), \quad \forall\, v \in Y$$

Petrov-Galerkin Methods



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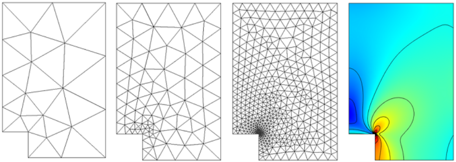

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❸ **Linear system:** Pick bases in X_h and Y_h.

$$\text{Seek} \quad \mathbf{u} \in \mathbb{C}^N \quad \text{such that} \quad \mathbf{A}\mathbf{u} = \mathbf{b} \qquad (\Lambda_h \text{ synthesis operator } \mathbf{u} \mapsto u_h)$$

Induced Operators

(Continuous and Discrete Operators)

We identify $\mathsf{a} \in \mathcal{L}(X \times Y; \mathbb{C})$ and $\mathsf{A} \in \mathcal{L}(X; Y')$ through:

$$\langle \mathsf{A}u, v \rangle_{Y' \times Y} := a(u, v), \quad \forall\ u \in X,\ \forall\ v \in Y$$

and we identify $\mathsf{a} \in \mathcal{L}(X_h \times Y_h; \mathbb{C})$ and $\mathsf{A}_h \in \mathcal{L}(X_h; Y_h')$ through:

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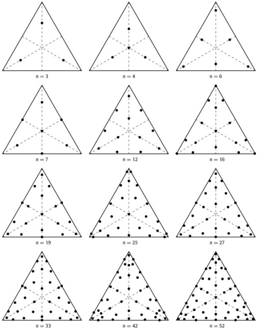
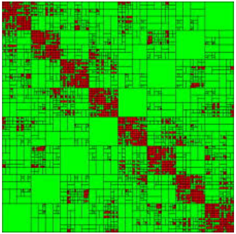
- **What about other variational crimes?**

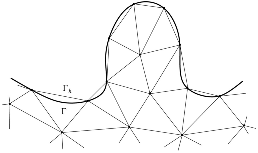
- Machine precision
- Quadrature rules
- Geometrical approximation error
- Fast methods (FMM, $\mathcal{H}$-matrices)



Quadrature rules



Hierarchical matrix



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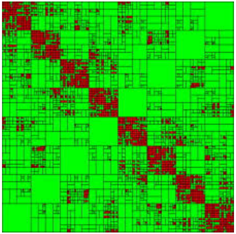

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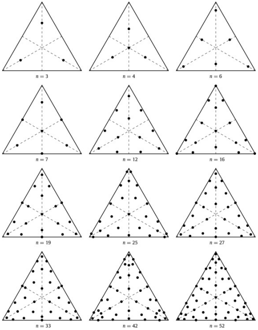

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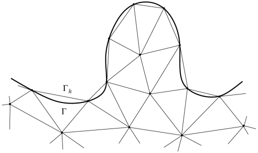

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Set $\nu \in [0,1)$ and let $u \in X$, $u_h \in X_h$ and $u_{h,\nu} \in X_h$ be the corresponding unique solutions. Then, if

$$\|\mathsf{A}_h - \mathsf{A}_{h,\nu}\|_{Y_h'} \leq \gamma_{\mathsf{A}} \nu \quad \text{and} \quad \|b_h - b_{h,\nu}\|_{Y_h'} \leq \nu \|b_h\|_{Y_h'}$$

it holds

$$\|u - u_h\|_X \leq (1 + \mathsf{K}_{\mathsf{A}}) \inf_{w_h \in X_h} \|u - w_h\|_X \qquad \text{(Cea's lemma)}$$

$$\|u - u_{h,\nu}\|_X \leq (1 + \mathsf{K}_{\mathsf{A}}) \left(1 + \frac{\mathsf{K}_{\mathsf{A}}}{1-\nu}\right) \inf_{w_h \in X_h} \|u - w_h\|_X + \frac{2\nu}{\gamma_{\mathsf{A}}(1-\nu)} \|b_h\|_{Y_h'}$$

[1] Escapil-Inchauspé and J-H, *Bi-Operator Preconditioning*, Computers & Mathematics with Applications, 2021.

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$$\|u - u_{h,\nu}\|_X \leq (1 + \mathsf{K}_\mathsf{A}) \left(1 + \frac{\mathsf{K}_\mathsf{A}}{1-\nu}\right) \inf_{w_h \in X_h} \|u - w_h\|_X + \frac{2\nu}{\gamma_\mathsf{A}(1-\nu)} \|b_h\|_{Y_h'}$$

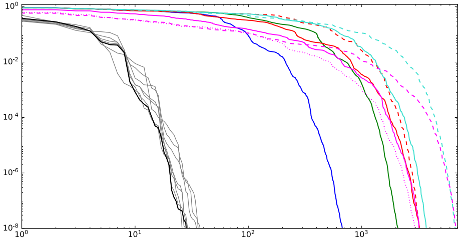
For small ν:

- Quasi-optimality constant $(1 + \mathsf{K}_\mathsf{A})^2$
- $\mathcal{O}(\nu)$-errors induced by A_ν and b_ν (e.g., $\nu = \mathcal{O}(h^{p+1})$)

[1] Escapil-Inchauspé and J-H, *Bi-Operator Preconditioning*, Computers & Mathematics with Applications, 2021.

We solve $\mathbf{Au} = \mathbf{b}$.

- Set $\mathbf{u}_0$ and seek $(\mathbf{u}_k)_{k=1}^N$ such that $\mathbf{u}_k \to \mathbf{u}$
- Define $\mathbf{r}_k := \mathbf{Au}_k - \mathbf{b}$ for $0 \leq k \leq N$
- SPD or HPD matrix $\Rightarrow$ Conjugate Gradient (e.g., Laplace: $-\Delta$)
- Indefinite matrix $\Rightarrow$ GMRES or GMRES(m) (e.g., Helmholtz/Maxwell)



Relative l^2-residual norm error vs. iteration count k

What about Iterative Solvers?

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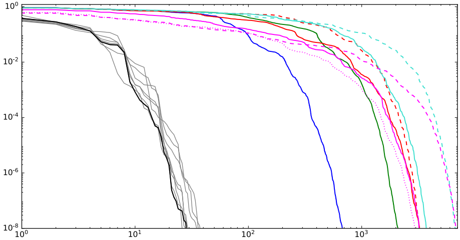

Relative l^2-residual norm error vs. iteration count k

Goal: $\frac{\|\mathbf{r}_k\|}{\|\mathbf{r}_0\|} \to 0$ as fast as possible (and h-independently)

CG: Condition number and spectral properties

- For HPD matrices, convergence for CG depends on the spectral condition number:

$$\kappa_S(\mathbf{A}) := \frac{|\lambda_{\max}(\mathbf{A})|}{|\lambda_{\min}(\mathbf{A})|}$$

- We apply CG to $\mathbf{Au} = \mathbf{b}$, para $\mathbf{u}_0 = \mathbf{b} \Rightarrow (\mathbf{u}_k)_{k\geq 1}$.

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Lemma (Linear bound for CG)

The residual error of CG at iteration $1 \leq k \leq N$ *yields:*

$$\frac{\|\mathbf{u}_k - \mathbf{u}\|_A}{\|\mathbf{u}_0 - \mathbf{u}\|_A} \leq \left(1 - \frac{1}{\sqrt{\kappa_S(\mathbf{A})}}\right)^k =: \varrho^k$$

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- For Laplace $\kappa_S(\mathbf{A}) = \mathcal{O}(h^{-2})$
- h-dependence comes from a mismatch between functional spaces : operator A is not an endomorphism
- When $h \to 0$: $\kappa_S(\mathbf{A}) \to \infty$, and $\varrho \to 1$.

Solution: Preconditioning (why we are here!)

We seek to solve

$$\mathbf{Au} = \mathbf{b}$$

Seek $\mathbf{P}$ such that:

1. $\mathbf{P}$ is relatively cheap to compute
2. $\mathbf{PA} \approx \mathbf{I}$ or iterative solvers perform better than on the original system

$$\text{Seek} \quad \mathbf{u} \in \mathbb{C}^N \quad \text{such that} \quad \mathbf{PAu} = \mathbf{Pb}$$

(Continuous problem)

For X, Y reflexive Banach spaces, $\mathsf{A} \in \mathcal{L}(X; Y')$ with norm $\|\mathsf{a}\|$, $b \in Y'$:

$$\text{Seek} \quad u \in X \quad \text{such that} \quad \mathsf{A}u = b$$

[2] R. Hiptmair, *Operator Preconditioning*, Computers and Mathematics with Applications, vol. 52, 2006.

Operator Preconditioning : PG Setting

(Continuous problem)

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$$\text{Seek} \quad u \in X \quad \text{such that} \quad \mathsf{A}u = b$$

- We use the OP framework[2] and introduce bounded linear operators $\mathsf{C}, \mathsf{N}, \mathsf{M}$ such that:

$$\begin{array}{ccc} X & \xrightarrow{\ \mathsf{A}\ } & Y' \\ {\scriptstyle \mathsf{M}^{-1}}\uparrow & & \downarrow{\scriptstyle \mathsf{N}^{-1}} \\ W' & \xleftarrow[\ \mathsf{C}\ ]{} & V \end{array}$$

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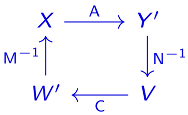

- Set $\mathsf{P} := \mathsf{M}^{-1}\mathsf{C}\mathsf{N}^{-1}$ and $\mathsf{PA} \in \mathcal{L}(X; X)$

Problem (OP-PG)

$$\textit{Seek} \quad \mathbf{u} \in \mathbb{C}^N \quad \textit{such that} \quad \mathbf{PAu} = \mathbf{Pb}$$

- Bubnov-Galerkin $Y = X$, $V = W$, $\mathsf{N} = \mathsf{M}^*$
- Opposite-order Preconditioning $V = Y'$, $W = X'$, $\mathsf{N} = \mathsf{M} = \mathrm{Id}$

[2] R. Hiptmair, *Operator Preconditioning*, Computers and Mathematics with Applications, vol. 52, 2006.

Operator Preconditioning : PG Setting

Theorem (Estimates for OP-PG[1])

Consider OP-PG. There holds that:

$$\kappa_S(\mathbf{PA}) \leq \frac{\|\mathsf{m}\|\|\mathsf{n}\|\|\mathsf{c}\|\|\mathsf{a}\|}{\gamma_\mathsf{M}\gamma_\mathsf{N}\gamma_\mathsf{C}\gamma_\mathsf{A}} =: \mathsf{K}_\star$$

Furthermore, the Euclidean condition number satisfies

$$\kappa_2(\mathbf{PA}) \leq \mathsf{K}_\star\, \mathsf{K}^2_{\Lambda_h} \quad (\Lambda_h \textit{ synthesis operator})$$

with

$$\mathsf{K}_{\Lambda_h} := \frac{\sup_{u_h \in X_h \setminus \{\mathbf{0}\}} \frac{\|u_h\|_X}{\|\mathbf{u}\|_2}}{\inf_{u_h \in X_h \setminus \{\mathbf{0}\}} \frac{\|u_h\|_X}{\|\mathbf{u}\|_2}} \quad \left(\textit{for } X = H^s(D) \textit{ then } \mathsf{K}_{\Lambda_h} \leq C \left(\frac{h_{max}}{h_{min}}\right)^{d/2} h_{min}^{-|s|} \right)$$

[1] P. Escapil-Inchauspé and C. Jerez-Hanckes, *Bi-Operator Preconditioning*, Computers & Mathematics with Applications, 2021.

Outline

Main ideas: Efficient Operator Preconditioning Schemes

- Apply Operator Preconditioning
- Use different precisions/tolerances for the preconditioner and stiffness matrix

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Iterative solvers are robust with respect to operator perturbations

Bi-Parametric Operator Preconditioning

For $\mu, \nu \in [0, 1)$, introduce suitably defined C_μ, A_ν, and b_ν.

$$
\begin{array}{ccc}
X & \xrightarrow{\mathsf{A}_\nu} & Y' \\
\mathsf{M}^{-1}\big\uparrow & & \big\downarrow \mathsf{N}^{-1} \\
W' & \xleftarrow[\mathsf{C}_\mu]{} & V
\end{array}
$$

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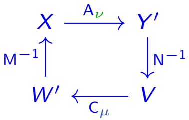


- There holds that $\mathsf{P}_\mu \mathsf{A}_\nu := (\mathsf{M}^{-1}\mathsf{C}_\mu \mathsf{N}^{-1})\mathsf{A}_\nu \in \mathcal{L}(X; X)$.
- One arrives at problem

Problem (OP-PG)

Seek $\mathbf{u}_\nu \in \mathbb{C}^N$ *such that* $\mathbf{P}_\mu \mathbf{A}_\nu \mathbf{u}_\nu = \mathbf{P}_\mu \mathbf{b}_\nu$

Theorem (Bi-Parametric Operator Preconditioning[1])

Consider OP-PG. For the spectral conditioning number, it holds that

$$\kappa_S(\mathbf{P}_\mu \mathbf{A}_\nu) \leq \mathsf{K}_\star \left(\frac{1+\mu}{1-\mu}\right)\left(\frac{1+\nu}{1-\nu}\right) =: \mathsf{K}_{\star,\mu,\nu}$$

and for the Euclidean version

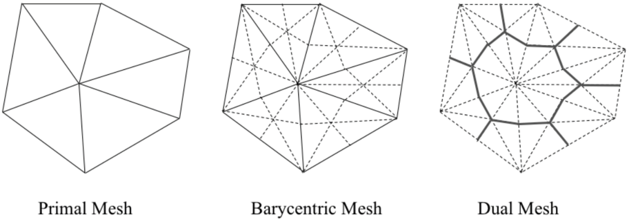
$$\kappa_2(\mathbf{P}_\mu \mathbf{A}_\nu) \leq \mathsf{K}_{\star,\mu,\nu}\, \mathsf{K}^2_{\Lambda_h}$$

- ✓ Controlled condition numbers w.r.t. h, μ, ν
- ✓ No cross-terms in μ and ν
- ✓ Bounded $\mu, \nu \Rightarrow$ bounded $\kappa_S(\mathbf{P}_\mu \mathbf{A}_\nu)$

[1] P. Escapil-Inchauspé and C. Jerez-Hanckes, *Bi-Operator Preconditioning*, Computers & Mathematics with Applications, 2021.

Application: Acoustic and EM time-harmonic wave scattering

- Traditionally solved via boundary integral equations
- Operator preconditioner in the form of Calderón (opposite order) preconditioning
- Preconditioner built via dual mesh
- Dual mesh is achieved via barycentric refinement leading to expensive computational costs



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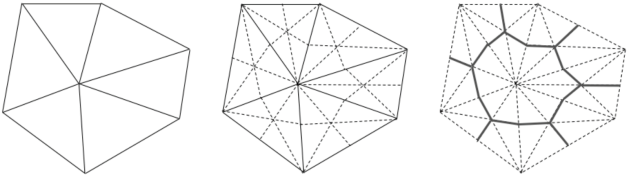
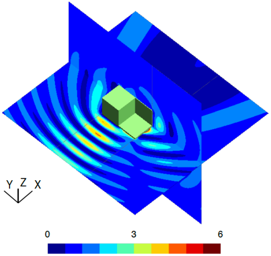
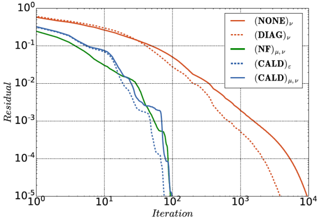


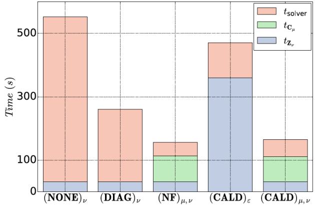
Primal Mesh Barycentric Mesh Dual Mesh

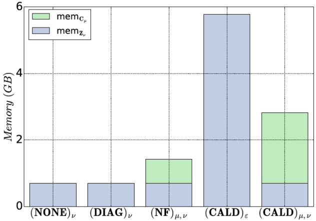
- Solution? Coarse approximation of **P**

EM Fichera cube with $k = 10$, $r = 10$, $N = 16113$, $N_b = 96678$, GMRES(200)
$\nu = 10^{-5}$, $\mu = 10^{-1}$



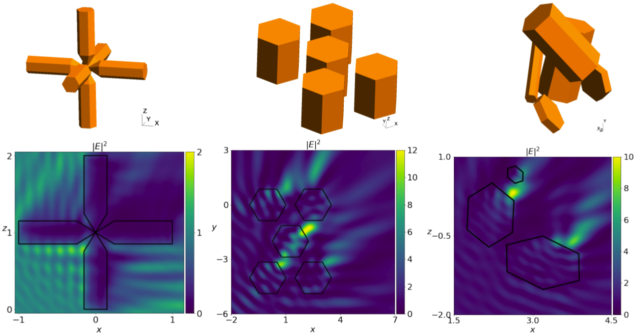






[2] Fast Calderón preconditioning for the Electric Field Integral Equation, Escapil-Inchauspé and Jerez-Hanckes, *IEEE Transactions on Antennas and Propagation*, **(2019)**.

We combine reduced quadrature + ACA + near-field cutoff.



Squared magnitude of the electric field for scattering by multiple particles.

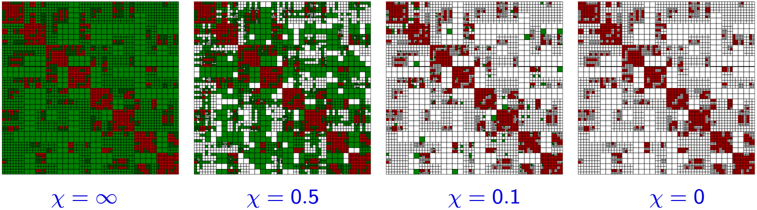
[3] Accelerated Calderón preconditioning for Maxwell transmission problems, Kleanthous, Betcke, Hewett, Escapil-Inchauspé, Jerez-Hanckes and Baran, *Journal of Computational Physics,* **(2022)**.

EM scattering by complex scatterers - Cutoff compression for Hmat[3]

For a cutoff $\chi \in [0, \infty]$, the admissible blocks X, Y for which

$$\text{dist}(X, Y) \leq \chi \tag{1}$$

are assembled using ACA, while all other admissible blocks are set to zero.



Block cluster trees for the EFIO on the unit cube with $k = 5$. Red: Inadmissible blocks. Green: Admissible blocks that require ACA. White: Admissible blocks that do not require assembly. As χ decreases from left to right the overall compression rate decreases, taking the values 0.83, 0.60, 0.17 and 0.14 respectively when the ACA parameter $\nu = 0.001$.

[3] Accelerated Calderón preconditioning for Maxwell transmission problems, Kleanthous, Betcke, Hewett, Escapil-Inchauspé, Jerez-Hanckes and Baran, *Journal of Computational Physics,* **(2022)**.

EM scattering by complex scatterers[3]

Applying our accelerated implementation to 3D electromagnetic scattering by an aggregate consisting of 8 monomer ice crystals of overall diameter 1cm at 664GHz leads to a 99% reduction in memory cost and at least a 75% reduction in total computation time compared to a non-accelerated implementation.

[3] Accelerated Calderón preconditioning for Maxwell transmission problems, Kleanthous, Betcke, Hewett, Escapil-Inchauspé, J-H and Baran, *Journal of Computational Physics,* **(2022)**.

Outline

Convergence bounds for iterative solvers?

Hilbert space setting

- Symmetric case: bounded $\kappa_S \Rightarrow$ Bounded linear convergence rate ✓
- Indefinite case: We need more information...

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Hilbert space setting

- Symmetric case: bounded $\kappa_S \Rightarrow$ Bounded linear convergence rate ✓
- Indefinite case: We need more information...
- $X \equiv H$ with H Hilbert space with

$$\forall\ u_h, v_h \in X_h, \quad (u_h, v_h)_H = (\mathbf{Hu}, \mathbf{v})_2 =: (\mathbf{u}, \mathbf{v})_H$$

- Matrix H-FoV of $\mathbf{Q} \in \mathbb{C}^{N \times N}$

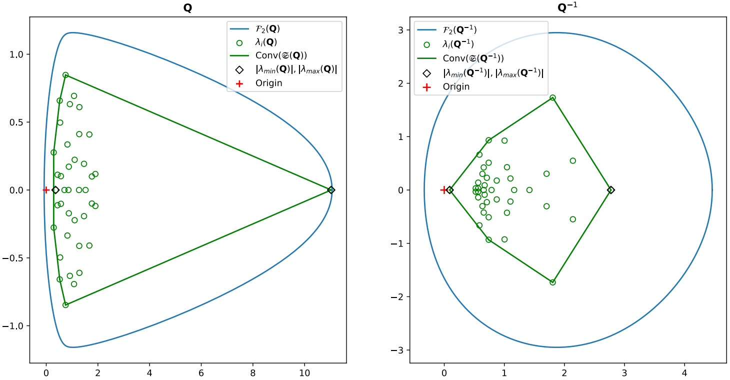
$$\mathcal{F}_H(\mathbf{Q}) := \left\{ \frac{(\mathbf{Qu}, \mathbf{u})_H}{(\mathbf{u}, \mathbf{u})_H} : \mathbf{u} \in \mathbb{C}^N \setminus \{\mathbf{0}\} \right\}$$

- Distance of $\mathcal{F}_H(\mathbf{Q})$ from the origin

$$\mathcal{V}_H(\mathbf{Q}) := \min_{z \in \mathcal{F}_H(\mathbf{Q})} |z| = \min_{\mathbf{u} \in \mathbb{C}^N \setminus \{\mathbf{0}\}} \frac{|(\mathbf{Qu}, \mathbf{u})_H|}{(\mathbf{u}, \mathbf{u})_H}$$

- Discrete H-FoV for Q_h and $\mathcal{V}_H(\mathsf{Q}_h)$ defined in the same fashion for any $\mathsf{Q}_h : X_h \to X_h$.

Example of 2-FoV



2-FoV boundary (blue line), eigenvalues (green circles), convex hull for eigenvalues (green line) and $|\lambda_{\min}|, |\lambda_{\max}|$ (black diamonds) for a matrix $\mathbf{Q} := \mathbf{I} + 0.5\mathbf{E} \in \mathbb{R}^{40\times 40}$ (left) and its inverse $\mathbf{Q}^{-1}$ (right).

GMRES(m)

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GMRES(m)

Convergence bounds for iterative solvers?

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- Symmetric case: bounded $\kappa_S \Rightarrow$ Bounded linear convergence rate ✓
- Indefinite case: We need more information...

- Application of the weighted (resp. Euclidean) restarted GMRES(m) to $\mathbf{Q}\mathbf{x} = \mathbf{d}$.
- We arrive at the residuals:

$$\|\mathbf{r}_k\|_H := \|\mathbf{d} - \mathbf{Q}\mathbf{x}_k\|_H = \min_{\mathbf{x}\in\mathcal{K}^k(\mathbf{Q},\mathbf{r}_0)} \|\mathbf{d} - \mathbf{Q}\mathbf{x}\|_H,$$

$$\|\tilde{\mathbf{r}}_k\|_2 := \|\mathbf{d} - \mathbf{Q}\tilde{\mathbf{x}}_k\|_2 = \min_{\mathbf{x}\in\mathcal{K}^k(\mathbf{Q},\mathbf{r}_0)} \|\mathbf{d} - \mathbf{Q}\mathbf{x}\|_2$$

GMRES(m)

Lemma (Weighted GMRES(m): Linear bounds[1])

Let $\mathbf{Q} \in \mathbb{C}^N$, *with* $0 < \mathcal{V}_H(\mathbf{Q})$ *and set* $1 \leq m \leq N$. *Then, the* k*-th residual of weighted GMRES*(m) *for* $1 \leq k \leq N$ *satisfies:*

$$\frac{\|\mathbf{r}_k\|_H}{\|\mathbf{r}_0\|_H} \leq \left(1 - \mathcal{V}_H(\mathbf{Q})\mathcal{V}_H\left(\mathbf{Q}^{-1}\right)\right)^{\frac{k}{2}}$$

with

$$\mathcal{V}_H(\mathbf{Q}) := \min_{\mathbf{u} \in \mathbb{C}^N \setminus \{\mathbf{0}\}} \frac{|(\mathbf{Q}\mathbf{u}, \mathbf{u})_H|}{(\mathbf{u}, \mathbf{u})_H}$$

[1] P. Escapil-Inchauspé and C. Jerez-Hanckes, *Bi-Operator Preconditioning*, Computers & Mathematics with Applications, 2021.

GMRES(m) for OP-PG

$$\Theta_k^{(m)} := \left(\frac{\|\mathbf{P}_\mu \mathbf{r}_k\|_H}{\|\mathbf{P}_\mu \mathbf{r}_0\|_H} \right)^{1/k} \quad \text{and} \quad \widetilde{\Theta}_k^{(m)} := \left(\frac{\|\mathbf{P}_\mu \tilde{\mathbf{r}}_k\|_2}{\|\mathbf{P}_\mu \mathbf{r}_0\|_2} \right)^{1/k}$$

Theorem (GMRES(m): Linear convergence estimates for OP-PG[1])

For OP-PG with inner product $(\cdot,\cdot)_H$, we assume that $\mathsf{P}_h\mathsf{A}_h$ and its inverse satisfy

$$\frac{\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}}{\|\mathsf{m}\|\|\mathsf{n}\|} \leq \mathcal{V}_H(\mathsf{P}_h\mathsf{A}_h) \quad \textit{and} \quad \frac{\gamma_{\mathsf{M}}\gamma_{\mathsf{N}}}{\|\mathsf{c}\|\|\mathsf{a}\|} \leq \mathcal{V}_H((\mathsf{P}_h\mathsf{A}_h)^{-1})$$

Then, GMRES(m) for $1 \leq k, m \leq N$ leads to

$$\Theta_k^{(m)} \leq \left(1 - \frac{1}{\mathsf{K}_\star}\right)^{\frac{1}{2}} \quad \text{and} \quad \widetilde{\Theta}_k^{(m)} \leq \mathsf{K}_{\Lambda_h} \left(1 - \frac{1}{\mathsf{K}_\star}\right)^{\frac{1}{2}}$$

- ✓ h-independent convergence for weighted GMRES(m)
- ✓ Offset factor K_{Λ_h} for Euclidean GMRES(m)

[1] P. Escapil-Inchauspé and C. Jerez-Hanckes, *Bi-Operator Preconditioning*, Computers & Mathematics with Applications, 2021.

GMRES(m) for Bi-Parametric OP-PG

Theorem (GMRES(m): Linear convergence estimates for Bi-Parametric OP-PG)

For Bi-Parametric OP-PG with inner product $(\cdot,\cdot)_H$, we assume that $\mathsf{P}_{h,\mu}\mathsf{A}_{h,\nu}$ and its inverse satisfy

$$\frac{\gamma_{\mathsf{C}_\mu}\gamma_{\mathsf{A}_\nu}}{\|\mathsf{m}\|\|\mathsf{n}\|} \leq \mathcal{V}_H(\mathsf{P}_{h,\mu}\mathsf{A}_{h,\nu}) \quad \text{and} \quad \frac{\gamma_{\mathsf{M}}\gamma_{\mathsf{N}}}{\|\mathsf{c}_\mu\|\|\mathsf{a}_\nu\|} \leq \mathcal{V}_H((\mathsf{P}_{h,\mu}\mathsf{A}_{h,\nu})^{-1})$$

Then, GMRES(m) for $1 \leq k, m \leq N$ leads to

$$\Theta_k^{(m)} \leq \left(1 - \frac{1}{\mathsf{K}_{\star,\mu,\nu}}\right)^{\frac{1}{2}} \quad \text{and} \quad \widetilde{\Theta}_k^{(m)} \leq \mathsf{K}_{\Lambda_h}\left(1 - \frac{1}{\mathsf{K}_{\star,\mu,\nu}}\right)^{\frac{1}{2}}$$

- ✓ Controlled convergence rates for GMRES(m) with respect to (μ,ν)-perturbations
- ✓ Bounded $\mu,\nu = \mathcal{O}(1)$ guarantee convergence for weighted GMRES(m) (and Euclidean GMRES(m) up to a K_{Λ_h}-term for $\mathsf{K}_{\Lambda_h} < 1$)

Super-linear convergence results

- *Carleman Class* of compact operators $C^p(H)$ for $p > 0$:

$$|||\mathsf{K}|||_p = \|\sigma(\mathsf{K})\|_p := \left(\sum_{i=1}^{\infty} \sigma_i(\mathsf{K})^p\right)^{1/p} < \infty \qquad (\sigma_i(K) \text{ singular values})$$

- Q is a *p-class Fredholm operator of the second-kind* if

$$\mathsf{Q} - \mathsf{I} =: \mathsf{K} \in \mathcal{C}^p(H) \quad \text{(compact)} \tag{2}$$

Define the commuting diagram

$$((\mathsf{CA}))^p_{\mu,\nu} : \quad \begin{array}{ccc} H & \xrightarrow{\mathsf{A}_\nu} & Y' \\ {\scriptstyle \mathsf{I}^{-1}}\uparrow & & \downarrow{\scriptstyle \mathsf{N}^{-1}} \\ H & \xleftarrow[\mathsf{C}_\mu]{} & V \end{array} .$$

We interested in the resulting equation $\mathsf{I}^{-1} C_\mu \mathsf{N}^{-1} \mathsf{A}_\nu u_{\mu,\nu} = b_\nu$

Super-linear convergence for $((\mathsf{CA}))^{p}_{\mu,\nu}$

Theorem (GMRES: Super-linear convergence estimates for $((\mathsf{CA}))^{p}_{\mu,\nu}$)

Consider $((\mathsf{CA}))^{p}_{\mu,\nu}$ *for any* $p \geq 0$ *and define* $\mathsf{K}_{\mu,\nu} := \mathsf{C}_{\mu}\mathsf{N}^{-1}\mathsf{A}_{\nu} - \mathsf{I} \in \mathcal{C}^{p}(H)$. *Then, for weighted and Euclidean GMRES, respectively, it holds that*

$$\begin{aligned} \Theta_k &\leq \frac{\|\mathsf{n}\|}{\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}\gamma_{\mathsf{M}}} \frac{\overline{\sigma}_k(\mathsf{K}_{\mu,\nu})}{(1-\mu)(1-\nu)} \qquad (\overline{\sigma}_k \textit{ arithmetic mean of } \sigma_k) \\ &\leq \frac{\|\mathsf{n}\|}{\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}\gamma_{\mathsf{M}}} \frac{|||\mathsf{K}_{\mu,\nu}|||_p}{(1-\mu)(1-\nu)} k^{-\frac{1}{p}} \quad \textit{if} \quad p > 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} \widetilde{\Theta}_k &\leq \mathsf{K}_{\Lambda_h} \frac{\|\mathsf{n}\|}{\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}\gamma_{\mathsf{M}}} \frac{\overline{\sigma}_k(\mathsf{K}_{\mu,\nu})}{(1-\mu)(1-\nu)} \\ &\leq \mathsf{K}_{\Lambda_h} \frac{\|\mathsf{n}\|}{\gamma_{\mathsf{C}}\gamma_{\mathsf{A}}\gamma_{\mathsf{M}}} \frac{|||\mathsf{K}_{\mu,\nu}|||_p}{(1-\mu)(1-\nu)} k^{-\frac{1}{p}} \quad \textit{if} \quad p > 0. \end{aligned} \tag{4}$$

- ✓ Weighted GMRES **converges super-linearly**
- ✓ Euclidean GMRES **can converge super-linearly** (e.g., for bounded K_{Λ_h})
- ✓ **Exhaustive** and **controlled** convergence results for GMRES

Discussion and key takeaways

1. For indefinite problems, the distribution of eigenvalues is not descriptive enough
2. Obtaining H-FoV robust formulations is non-trivial
3. Superlinear convergence bounds are valid for unrestarted GMRES

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Questions?

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References

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