Complex eigenvalues of real nonsymmetric matrices

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$$\mathbf{B} \neq \mathbf{B}^T \in \mathbb{R}^{m \times m}$$

can generally have complex eigenvalues: how many?

Less well known:
there exist $\mathbf{T} = \mathbf{T}^T \in \mathbb{R}^{m \times m}, \mathbf{W} = \mathbf{W}^T \in \mathbb{R}^{m \times m}$ with

$$\mathbf{B} = \mathbf{T}\mathbf{W}$$

If $\mathbf{A} = \mathbf{A}^T$: $inertia(\mathbf{A}) = (p, n, z)$ where $\mathbf{A}$ has $p$ositive, $n$egative, $z$ero eigenvalues

Parlett convention: symmetric letters for symmetric matrices, so

$$\mathbf{T} = \mathbf{T}^T, \mathbf{W} = \mathbf{W}^T \quad \text{but} \quad \mathbf{B} \neq \mathbf{B}^T$$

We will show number of non-real (complex) eigenvalues of $\mathbf{B}$ gives restrictions on $\mathbf{T}, \mathbf{W}$.

For simplicity assume $\mathbf{B}$ invertible $\Rightarrow \mathbf{T}, \mathbf{W}$ invertible

Note $\lambda > 0 \iff \frac{1}{\lambda} > 0$ so $inertia(\mathbf{A}) = inertia(\mathbf{A}^{-1})$

Suppose $\mathbf{B} = \mathbf{TW}$. (Note $\mathbf{T}^{-1}, \mathbf{W}^{-1}$ also real symmetric.)

$$
\begin{aligned}
\lambda \text{ is an eigenvalue of } \mathbf{B} \quad &\iff \quad \mathbf{B} - \lambda \mathbf{I} \quad \text{is singular} \\
&\iff \quad \mathbf{TW} - \lambda \mathbf{I} \quad \text{is singular} \\
&\iff \quad \mathbf{W} - \lambda \mathbf{T}^{-1} \quad \text{is singular.}
\end{aligned}
$$

Now $\mathbf{V}(\theta) = \theta \mathbf{W} + (1-\theta)\mathbf{T}^{-1}$ is real symmetric $\forall \theta \in \mathbb{R} \Rightarrow \mathbf{V}(\theta)$ has real eigenvalues that depend continuously on θ.

Lemma:

If $\mathbf{B} = \mathbf{TW}$ with $inertia(\mathbf{T}) \neq inertia(\mathbf{W})$ then $\mathbf{B}$ has at least one real negative eigenvalue.

$$\mathbf{V}(\theta) = \theta\mathbf{W} + (1-\theta)\mathbf{T}^{-1}$$

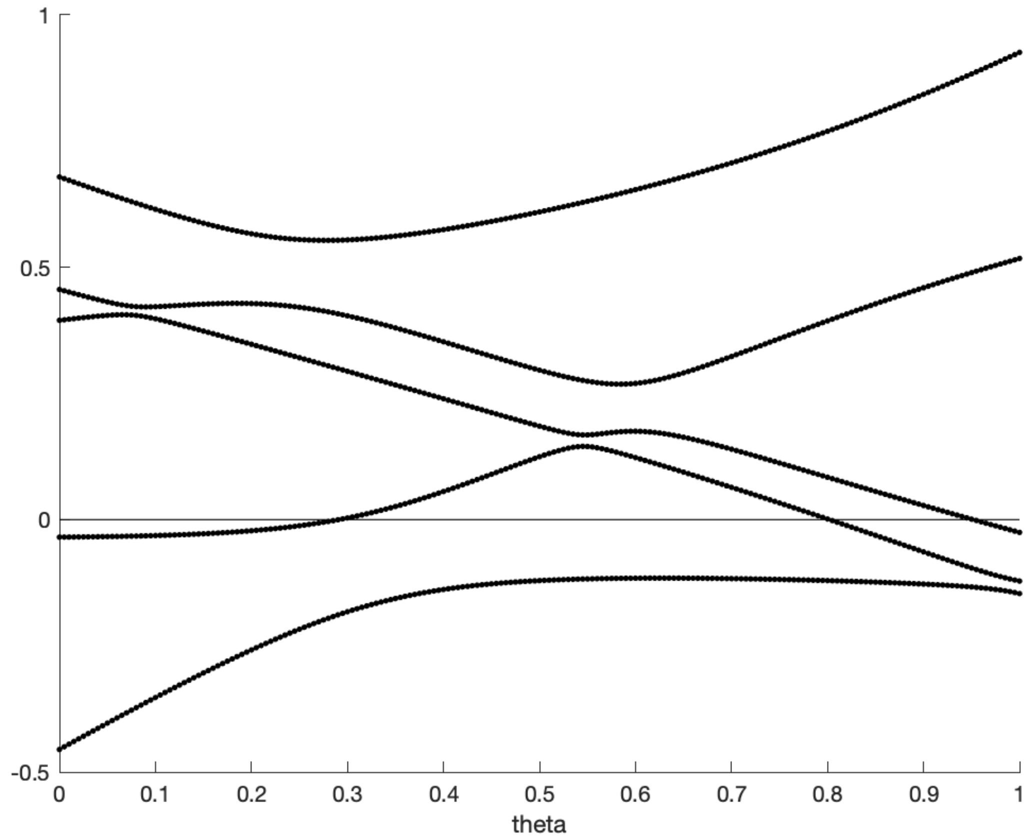
Lemma
If $\mathbf{B} = \mathbf{TW}$ with $inertia(\mathbf{T}) \neq inertia(\mathbf{W})$ then $\mathbf{B}$ has at least one real negative eigenvalue.

Proof:
$\mathbf{T}$, thus $\mathbf{T}^{-1} = \mathbf{V}(0)$ and $\mathbf{W} = \mathbf{V}(1)$ have different inertia $\Rightarrow$ at least one eigenvalue of $\mathbf{V}(\theta)$ changes sign as θ varies from 0 to 1. Intemediate Value Theorem $\Rightarrow \exists\widehat{\theta} \in (0,1)$ with $\mathbf{V}(\widehat{\theta}) = \widehat{\theta}\mathbf{W} + (1-\widehat{\theta})\mathbf{T}^{-1}$ is singular. Thus

$$\mathbf{W} + \frac{(1-\widehat{\theta}\,)}{\widehat{\theta}}\mathbf{T}^{-1} \quad \text{and so also} \quad \mathbf{TW} + \frac{(1-\widehat{\theta}\,)}{\widehat{\theta}}\mathbf{I}$$

are singular, hence $\frac{(\widehat{\theta}-1)}{\widehat{\theta}} < 0$ is an eigenvalue of $\mathbf{TW} = \mathbf{B}$
$\square$

Example



$$\mathbf{T}^{-1} = \begin{bmatrix} 0.33 & -.05 & -.29 & 0.01 & 0.01 \\ -.05 & 0.36 & -.11 & -.22 & -.19 \\ -.29 & -.11 & -.32 & 0.11 & -.01 \\ 0.01 & -.22 & 0.11 & 0.49 & -.12 \\ 0.01 & -.19 & -.01 & -.12 & 0.18 \end{bmatrix}, \mathbf{W} = \begin{bmatrix} 0.14 & 0.10 & 0.25 & 0.09 & -.28 \\ 0.10 & -.07 & 0.02 & 0.08 & -.11 \\ 0.25 & 0.02 & 0.49 & -.11 & -.23 \\ 0.09 & 0.08 & -.11 & 0.24 & -.34 \\ -.28 & -.11 & -.23 & -.34 & 0.35 \end{bmatrix}$$

From now on:

$$inertia(\mathbf{T}) = (p, n, 0), \quad inertia(\mathbf{W}) = (\widehat{p}, \widehat{n}, 0)$$

With a little more care one can prove:

$\mathbf{B} = \mathbf{TW}$ has at least $|p - \widehat{p}\,| = |n - \widehat{n}\,|$ real negative eigenvalues (counting multiplicities).

By similarly considering: $\mathbf{U}(\phi) = \phi\mathbf{W} + (1 - \phi)(-\mathbf{T}^{-1})$, noting $\mathbf{U}(1) = \mathbf{W}$ and $\mathbf{U}(0) = -\mathbf{T}^{-1}$ which has inertia $(n, p, 0)$ we have $\frac{1-\phi}{\phi} > 0$ is an eigenvalue of $\mathbf{B} = \mathbf{TW}$ when $\mathbf{U}(\phi)$ singular, thus

$\mathbf{B} = \mathbf{TW}$ has at least $|n - \widehat{p}\,| = |p - \widehat{n}\,|$ real and positive eigenvalues (counting multiplicities).

Main result:
If $\mathbf{B} = \mathbf{TW} \in \mathbb{R}^{m\times m}$ has $m - s$ non-real eigenvalues ($\frac{1}{2}m - \frac{1}{2}s$ complex conjugate pairs) then

$$\frac{1}{2}m - \frac{1}{2}s \leq p \leq \frac{1}{2}m + \frac{1}{2}s.$$

Proof (by contradiction) If $p < \frac{1}{2}m - \frac{1}{2}s$ then $n > \frac{1}{2}m + \frac{1}{2}s$ because $p + n = m$. Likewise if $p > \frac{1}{2}m + \frac{1}{2}s$ then $n < \frac{1}{2}m - \frac{1}{2}s$. Thus, whatever $\widehat{p}$

$$\underbrace{|p - \widehat{p}\,|}_{-ve} + \underbrace{|n - \widehat{p}\,|}_{+ve} > \frac{1}{2}m + \frac{1}{2}s - \left(\frac{1}{2}m - \frac{1}{2}s\right) = s.$$

□

Similarly
If $\mathbf{B} = \mathbf{TW} \in \mathbb{R}^{m\times m}$ has $m - s$ non-real eigenvalues then

$$\frac{1}{2}m - \frac{1}{2}s \leq \widehat{p} \leq \frac{1}{2}m + \frac{1}{2}s.$$

Thus if $\mathbf{B}$ has all non-real (complex) eigenvalues then (taking $s = 0$)

$$inertia(\mathbf{T}) = (\frac{m}{2}, \frac{m}{2}, 0) = inertia(\mathbf{W})$$

Note: converse *not* true: $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$ for any $\mathbf{A}$ whatever it's inertia

Existence of TW factorisation: Jordan form: $\mathbf{B} = \mathbf{S}\mathbf{J}\mathbf{S}^{-1}$
$\mathbf{J} =$ blockdiag of Jordan blocks

$$\widehat{\mathbf{J}} = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} \in \mathbb{C}^{\ell\times\ell}$$

$$\text{Note :}\quad \widehat{\mathbf{J}} = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix} \begin{bmatrix} & & & & \lambda \\ & & & \lambda & 1 \\ & & \iddots & \iddots & \\ & \lambda & 1 & & \\ \lambda & 1 & & & \end{bmatrix} = \widehat{\mathbf{Y}}\,\widehat{\mathbf{ℷ}}$$

So $\mathbf{J} = \mathbf{Y}\mathbf{ℷ}$

If all λ real then $\mathbf{S}$ real and

$$\mathbf{B} = \mathbf{S}\mathbf{J}\mathbf{S}^{-1} = \underbrace{\mathbf{S}\mathbf{Y}\mathbf{S}^{T}}_{\mathbf{T}} \underbrace{\mathbf{S}^{-T} \perp \mathbf{S}^{-1}}_{\mathbf{W}}$$

If $\lambda = a \pm ib$ then can use real Jordan form to likewise prove existence.

Postscript: some of these results were already known (but forgotten?):

Alexander Ostrowski, Uber produkte hermitescher matrizen und buschel hermitescher formen,
Mathematische Zeitschrift, 72 (1959), pp. 1–15

My manuscript:

Andy Wathen, On complex eigenvalues of a real nonsymmetric matrix,
math arXiv:2503.18501 (submitted to Electronic Journal on Linear Algebra)