Part 7: SQP methods for equality constrained optimization

Nick Gould (RAL)

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) \text{ subject to } c(x)=0$$

Part C course on continuoue optimization

EQUALITY CONSTRAINED MINIMIZATION

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \; f(x) \text{ subject to } c(x)=0$$

where the **objective function** $f : \mathbb{R}^n \longrightarrow \mathbb{R}$
and the **constraints** $c : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ $(m \leq n)$

- assume that $f,\; c \in C^1$ (sometimes C^2) and Lipschitz
- often in practice this assumption violated, but not necessary
- easily generalized to inequality constraints . . . but may be better to use interior-point methods for these

OPTIMALITY AND NEWTON'S METHOD

1st order optimality:

$$g(x,y) \equiv g(x) - A^T(x)y = 0 \text{ and } c(x)=0$$

nonlinear system (linear in y)
$\Longrightarrow$
use Newton's method to find a correction (s,w) to (x,y)
$\Longrightarrow$

$$\begin{pmatrix} H(x,y) & -A^T(x) \\ A(x) & 0 \end{pmatrix}\begin{pmatrix} s \\ w \end{pmatrix} = -\begin{pmatrix} g(x,y) \\ c(x) \end{pmatrix}$$

ALTERNATIVE FORMULATIONS

unsymmetric:

$$\begin{pmatrix} H(x,y) & -A^T(x) \\ A(x) & 0 \end{pmatrix}\begin{pmatrix} s \\ w \end{pmatrix} = -\begin{pmatrix} g(x,y) \\ c(x) \end{pmatrix}$$

or symmetric:

$$\begin{pmatrix} H(x,y) & A^T(x) \\ A(x) & 0 \end{pmatrix}\begin{pmatrix} s \\ -w \end{pmatrix} = -\begin{pmatrix} g(x,y) \\ c(x) \end{pmatrix}$$

or (with $y^+ = y + w$) unsymmetric:

$$\begin{pmatrix} H(x,y) & -A^T(x) \\ A(x) & 0 \end{pmatrix}\begin{pmatrix} s \\ y^+ \end{pmatrix} = -\begin{pmatrix} g(x) \\ c(x) \end{pmatrix}$$

or symmetric:

$$\begin{pmatrix} H(x,y) & A^T(x) \\ A(x) & 0 \end{pmatrix}\begin{pmatrix} s \\ -y^+ \end{pmatrix} = -\begin{pmatrix} g(x) \\ c(x) \end{pmatrix}$$

DETAILS

- Often approximate with symmetric $B \approx H(x,y) \Longrightarrow$ e.g.

$$\begin{pmatrix} B & A^T(x) \\ A(x) & 0 \end{pmatrix} \begin{pmatrix} s \\ -y^+ \end{pmatrix} = - \begin{pmatrix} g(x) \\ c(x) \end{pmatrix}$$

- solve system using
  - unsymmetric (LU) factorization of $\begin{pmatrix} B & -A^T(x) \\ A(x) & 0 \end{pmatrix}$
  - symmetric (indefinite) factorization of $\begin{pmatrix} B & A^T(x) \\ A(x) & 0 \end{pmatrix}$
  - symmetric factorizations of B and the Schur Complement $A(x)B^{-1}A^T(x)$
  - iterative method (GMRES(k), MINRES, CG within $\mathcal{N}(A)$,...)

SEQUENTIAL QUADRATIC PROGRAMMING - SQP

or **successive** quadratic programming
or **recursive** quadratic programming (RQP)

> Given (x_0, y_0), set $k = 0$
> Until "convergence" iterate:
> Compute a suitable symmetric B_k using (x_k, y_k)
> Find
> $$s_k = \arg\min_{s \in \mathbb{R}^n} g_k^T s + \tfrac{1}{2} s^T B_k s \ \text{ subject to } \ A_k s = -c_k$$
> along with associated Lagrange multiplier estimates y_{k+1}
> Set $x_{k+1} = x_k + s_k$ and increase k by 1

AN ALTERNATIVE INTERPRETATION

$$\textbf{QP}: \ \underset{s \in \mathbb{R}^n}{\text{minimize}} \ g(x)^T s + \tfrac{1}{2} s^T B s \ \text{ subject to } \ A(x)s = -c(x)$$

- QP = **quadratic program**
- first-order model of constraints $c(x+s)$
- second-order model of objective $f(x+s)$ ... but B includes curvature of constraints

solution to QP satisfies

$$\begin{pmatrix} B & A^T(x) \\ A(x) & 0 \end{pmatrix} \begin{pmatrix} s \\ -y^+ \end{pmatrix} = - \begin{pmatrix} g(x) \\ c(x) \end{pmatrix}$$

ADVANTAGES

- simple
- fast
  - quadratically convergent with $B_k = H(x_k, y_k)$
  - superlinearly convergent with good $B_k \approx H(x_k, y_k)$
    - don't actually need $B_k \longrightarrow H(x_k, y_k)$

PROBLEMS WITH PURE SQP

- how to choose B_k?
- what if QP$_k$ is unbounded from below? and when?
- how do we globalize this iteration?

QP SUB-PROBLEM

$$\underset{s \in \mathbb{R}^n}{\text{minimize}} \;\; g^T s + \tfrac{1}{2} s B s \;\; \text{subject to} \;\; As = -c$$

- ⊙ need constraints to be consistent
  - ⋄ OK if A is full rank
- ⊙ need B to be positive (semi-) definite when $As = 0$

  $\Longleftrightarrow$

  $N^T B N$ positive (semi-) definite where the columns of N form a basis for null(A)

  $\Longleftrightarrow$

  $$\begin{pmatrix} B & A^T \\ A & 0 \end{pmatrix}$$

  (is non-singular and) has m −ve eigenvalues

LINESEARCH SQP METHODS

$$s_k = \underset{s \in \mathbb{R}^n}{\arg\min} \; g_k^T s + \tfrac{1}{2} s^T B_k s \;\; \text{subject to} \;\; A_k s = -c_k$$

Basic idea:

- ⊙ Pick $x_{k+1} = x_k + \alpha_k s_k$, where
  - ⋄ α_k is chosen so that

    $$\Phi(x_k + \alpha_k s_k, p_k) \text{“}<\text{”} \Phi(x_k, p_k)$$

  - ⋄ $\Phi(x, p)$ is a “suitable” merit function
  - ⋄ p_k are parameters
- ⊙ vital that s_k is a descent direction for $\Phi(x, p_k)$ at x_k
- ⊙ normally require that B_k is positive definite

SUITABLE MERIT FUNCTIONS. I

The **quadratic penalty function**:

$$\Phi(x, \mu) = f(x) + \frac{1}{2\mu} \|c(x)\|_2^2$$

> **Theorem 7.1.** Suppose that B_k is positive definite, and that (s_k, y_{k+1}) are the SQP search direction and its associated Lagrange multiplier estimates for the problem
>
> $$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; f(x) \;\; \text{subject to} \;\; c(x) = 0$$
>
> at x_k. Then if x_k is not a first-order critical point, s_k is a descent direction for the quadratic penalty function $\Phi(x, \mu_k)$ at x_k whenever
>
> $$\mu_k \leq \frac{\|c(x_k)\|_2}{\|y_{k+1}\|_2}$$

PROOF OF THEOREM 7.1

SQP direction s_k and associated multiplier estimates y_{k+1} satisfy

$$B_k s_k - A_k^T y_{k+1} = -g_k \tag{1}$$

and

$$A_k s_k = -c_k. \tag{2}$$

$$(1) + (2) \Longrightarrow s_k^T g_k = -s_k^T B_k s_k + s_k^T A_k^T y_{k+1} = -s_k^T B_k s_k - c_k^T y_{k+1} \tag{3}$$

$$(2) \Longrightarrow \frac{1}{\mu_k} s_k^T A_k^T c_k = -\frac{\|c_k\|_2^2}{\mu_k}. \tag{4}$$

(3) + (4), the positive definiteness of B_k, the Cauchy-Schwarz inequality, the required bound on μ_k, and $s_k \neq 0$ if x_k is not critical $\Longrightarrow$

$$s_k^T \nabla_x \Phi(x_k) = s_k^T \left(g_k + \frac{1}{\mu_k} A_k^T c_k \right) = -s_k^T B_k s_k - c_k^T y_{k+1} - \frac{\|c_k\|_2^2}{\mu_k}$$

$$< -\|c_k\|_2 \left(\frac{\|c_k\|_2}{\mu_k} - \|y_{k+1}\|_2 \right) \leq 0$$

NON-DIFFERENTIABLE EXACT PENALTIES

The **non-differentiable exact penalty function**:

$$\Phi(x,\rho) = f(x) + \rho\|c(x)\|$$

for any norm $\|\cdot\|$ and scalar $\rho > 0$.

> **Theorem 7.2.** Suppose that $f, c \in C^2$, and that x_* is an isolated local minimizer of $f(x)$ subject to $c(x) = 0$, with corresponding Lagrange multipliers y_*. Then x_* is also an isolated local minimizer of $\Phi(x,\rho)$ provided that
>
> $$\rho > \|y_*\|_D,$$
>
> where the **dual norm**
>
> $$\|y\|_D = \sup_{x\neq 0} \frac{y^T x}{\|x\|}.$$

SUITABLE MERIT FUNCTIONS. II

The non-differentiable exact penalty function:

$$\Phi(x,\rho) = f(x) + \rho\|c(x)\|$$

for any norm $\|\cdot\|$ (with dual norm $\|\cdot\|_D$) and scalar $\rho > 0$.

> **Theorem 7.3.** Suppose that B_k is positive definite, and that (s_k, y_{k+1}) are the SQP search direction and its associated Lagrange multiplier estimates for the problem
>
> $$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \ f(x)\ \ \text{subject to}\ \ c(x) = 0$$
>
> at x_k. Then if x_k is not a first-order critical point, s_k is a descent direction for the non-differentiable penalty function $\Phi(x,\rho_k)$ at x_k whenever $\rho_k \geq \|y_{k+1}\|_D$

PROOF OF THEOREM 7.3

Taylor's theorem applied to f and c + (2) $\Longrightarrow$ (for small α)

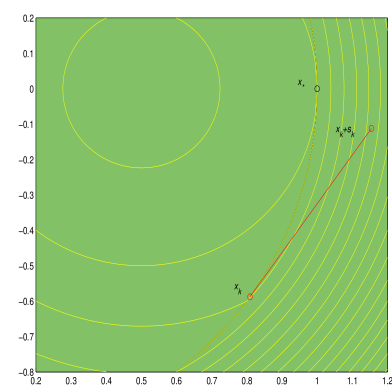
$$\begin{aligned}\Phi(x_k+\alpha s_k,\rho_k) - \Phi(x_k,\rho_k) &= \alpha s_k^T g_k + \rho_k\left(\|c_k + \alpha A_k s_k\| - \|c_k\|\right) + O(\alpha^2)\\ &= \alpha s_k^T g_k + \rho_k\left(\|(1-\alpha)c_k\| - \|c_k\|\right) + O(\alpha^2)\\ &= \alpha\left(s_k^T g_k - \rho_k\|c_k\|\right) + O\left(\alpha^2\right)\end{aligned}$$

+ (3), the positive definiteness of B_k, the Hölder inequality, and $s_k \neq 0$ if x_k is not critical $\Longrightarrow$

$$\begin{aligned}\Phi(x_k+\alpha s_k,\rho_k) - \Phi(x_k,\rho_k) &= -\alpha\left(s_k^T B_k s_k + c_k^T y_{k+1} + \rho_k\|c_k\|\right) + O(\alpha^2)\\ &< -\alpha\left(-\|c_k\|\|y_{k+1}\|_D + \rho_k\|c_k\|\right) + O(\alpha^2)\\ &= -\alpha\|c_k\|\left(\rho_k - \|y_{k+1}\|_D\right) + O(\alpha^2) < 0\end{aligned}$$

because of the required bound on ρ_k, for sufficiently small α. Hence sufficiently small steps along s_k from non-critical x_k reduce $\Phi(x,\rho_k)$.

THE MARATOS EFFECT



ℓ_1 non-differentiable exact penalty function ($\rho = 1$): $f(x) = 2(x_1^2 + x_2^2 - 1) - x_1$ and $c(x) = x_1^2 + x_2^2 - 1$ solution: $x_* = (1,0)$, $y_* = \frac{3}{2}$

Maratos effect: merit function may prevent acceptance of the SQP step arbitrarily close to $x_* \Longrightarrow$ slow convergence

AVOIDING THE MARATOS EFFECT

The Maratos effect occurs because the curvature of the constraints is not adequately represented by linearization in the SQP model:

$$c(x_k + s_k) = O(\|s_k\|^2)$$

$\Longrightarrow$ need to correct for this curvature

$\Longrightarrow$ use a **second-order correction** from $x_k + s_k$:

$$c(x_k + s_k + s_k^{\rm C}) = o(\|s_k\|^2)$$

also do not want to destroy potential for fast convergence $\Longrightarrow$

$$s_k^{\rm C} = o(s_k)$$

POPULAR 2ND-ORDER CORRECTIONS

- ⊙ minimum norm solution to $c(x_k + s_k) + A(x_k + s_k)s_k^{\rm C} = 0$

$$\begin{pmatrix} I & A^T(x_k+s_k) \\ A(x_k+s_k) & 0 \end{pmatrix} \begin{pmatrix} s_k^{\rm C} \\ -y_{k+1}^{\rm C} \end{pmatrix} = - \begin{pmatrix} 0 \\ c(x_k+s_k) \end{pmatrix}$$

- ⊙ minimum norm solution to $c(x_k + s_k) + A(x_k)s_k^{\rm C} = 0$

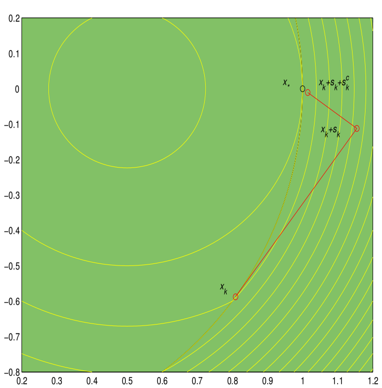
$$\begin{pmatrix} I & A^T(x_k) \\ A(x_k) & 0 \end{pmatrix} \begin{pmatrix} s_k^{\rm C} \\ -y_{k+1}^{\rm C} \end{pmatrix} = - \begin{pmatrix} 0 \\ c(x_k+s_k) \end{pmatrix}$$

- ⊙ another SQP step from $x_k + s_k$

$$\begin{pmatrix} H(x_k+s_k, y_k^+) & A^T(x_k+s_k) \\ A(x_k+s_k) & 0 \end{pmatrix} \begin{pmatrix} s_k^{\rm C} \\ -y_{k+1}^{\rm C} \end{pmatrix} = - \begin{pmatrix} g(x_k+s_k) \\ c(x_k+s_k) \end{pmatrix}$$

- ⊙ etc., etc.

2ND-ORDER CORRECTIONS IN ACTION



ℓ_1 non-differentiable exact penalty function ($\rho = 1$): $f(x) = 2(x_1^2 + x_2^2 - 1) - x_1$ and $c(x) = x_1^2 + x_2^2 - 1$ solution: $x_* = (1, 0)$, $y_* = \frac{3}{2}$

- ⊙ (very) fast convergence
- ⊙ $x_k + s_k + s_k^{\rm C}$ reduces $\Phi \Longrightarrow$ global convergence

TRUST-REGION SQP METHODS

Obvious trust-region approach:

$$s_k = \arg\min_{s \in \mathbb{R}^n} g_k^T s + \tfrac{1}{2} s^T B_k s \;\text{ subject to }\; A_k s = -c_k \;\text{ and }\; \|s\| \le \Delta_k$$

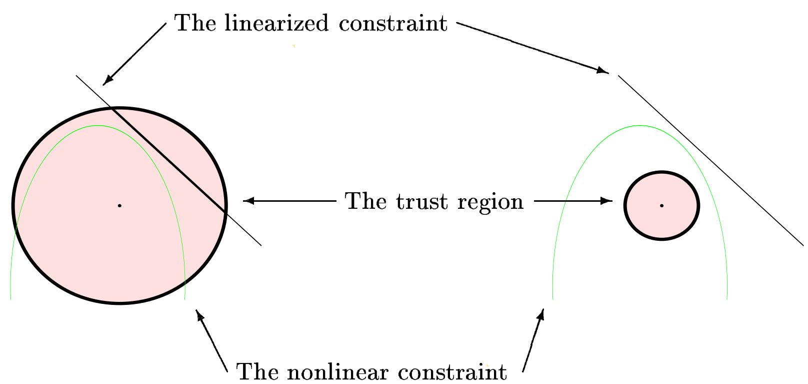
- ⊙ do not require that B_k be positive definite
  $\Longrightarrow$ can use $B_k = H(x_k, y_k)$
- ⊙ if $\Delta_k < \Delta^{\rm CRIT}$ where

$$\Delta^{\rm CRIT} \stackrel{\rm def}{=} \min \|s\| \;\text{ subject to }\; A_k s = -c_k$$

$\Longrightarrow$ **no solution to trust-region subproblem**

$\Longrightarrow$ simple trust-region approach to SQP is flawed if $c_k \neq 0 \Longrightarrow$ need to consider alternatives

INFEASIBILITY OF THE SQP STEP



ALTERNATIVES

- ⊙ the S$\ell_\mathbf{p}$QP method of Fletcher
- ⊙ composite step SQP methods
  - ⋄ constraint relaxation (Vardi)
  - ⋄ constraint reduction (Byrd–Omojokun)
  - ⋄ constraint lumping (Celis–Dennis–Tapia)
- ⊙ the filter-SQP approach of Fletcher and Leyffer

THE Sℓ_pQP METHOD

Try to minimize the ℓ_p-(exact) penalty function

$$\Phi(x,\rho) = f(x) + \rho\|c(x)\|_p$$

for sufficiently large $\rho > 0$ and some ℓ_p norm $(1 \leq p \leq \infty)$, using a trust-region approach

Suitable model problem: $\ell_\mathbf{p}$**QP**

$$\underset{s\in\mathbb{R}^n}{\text{minimize}} \;\; (f_k+)\; g_k^T s + \tfrac{1}{2}s^T B_k s + \rho\|c_k + A_k s\|_p \;\; \text{subject to} \;\; \|s\| \leq \Delta_k$$

- ⊙ model problem always consistent
- ⊙ when ρ and Δ_k are large enough, model minimizer = SQP direction
- ⊙ when the norms are polyhedral (e.g., ℓ_1 or ℓ_∞ norms), $\ell_\mathbf{p}$QP is equivalent to a quadratic program . . .

THE ℓ_1QP SUBPROBLEM

$\ell_\mathbf{1}$QP model problem with an ℓ_∞ trust region

$$\underset{s\in\mathbb{R}^n}{\text{minimize}} \;\; g_k^T s + \tfrac{1}{2}s^T B_k s + \rho\|c_k + A_k s\|_1 \;\; \text{subject to} \;\; \|s\|_\infty \leq \Delta_k$$

But

$$c_k + A_k s = u - v, \quad \text{where} \;\; (u,v) \geq 0$$

$\Longrightarrow$ $\ell_\mathbf{1}$QP equivalent to quadratic program (QP):

$$\begin{array}{rl} \underset{s\in\mathbb{R}^n,\, u,v\in\mathbb{R}^m}{\text{minimize}} & g_k^T s + \tfrac{1}{2}s^T B_k s + \rho(e^T u + e^T v) \\ \text{subject to} & A_k s - u + v = -c_k \\ & u \geq 0, \;\; v \geq 0 \\ \text{and} & -\Delta_k e \leq s \leq \Delta_k e \end{array}$$

- ⊙ good methods for solving QP
- ⊙ can exploit structure of u and v variables

PRACTICAL Sℓ_1QP METHODS

- Cauchy point requires solution to ℓ_1LP model:

$$\underset{s\in\mathbb{R}^n}{\text{minimize}}\ g_k^T s + \rho\|c_k + A_k s\|_1 \ \text{subject to}\ \|s\|_\infty \le \Delta_k$$

- approximate solutions to both ℓ_1LP and ℓ_1QP subproblems suffice
- need to adjust ρ as method progresses
- easy to generalize to inequality constraints
- globally convergent, but needs second-order correction for fast asymptotic convergence
- if $c(x) = 0$ are inconsistent, converges to (locally) least value of infeasibility $\|c(x)\|$

COMPOSITE-STEP METHODS

Aim: find **composite step**

$$s_k = n_k + t_k$$

where

the **normal step** n_k moves towards feasibility of the linearized constraints (within the trust region)

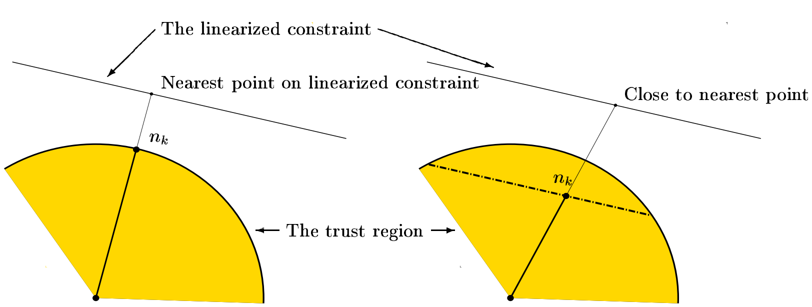
$$\|A_k n_k + c_k\| < \|c_k\|$$

(model objective may get worse)

the **tangential step** t_k reduces the model objective function (within the trust-region) without sacrificing feasibility obtained from n_k

$$A_k(n_k + t_k) = A_k n_k \implies A_k t_k = 0$$

NORMAL AND TANGENTIAL STEPS



Points on dotted line are all potential tangential steps

CONSTRAINT RELAXATION — VARDI

normal step: relax

$$A_k s = -c_k \ \text{and}\ \|s\| \le \Delta_k$$

to

$$A_k n = -\sigma_k c_k \ \text{and}\ \|n\| \le \Delta_k$$

where $\sigma_k \in [0,1]$ is small enough so that there is a feasible n_k

tangential step:

$$\text{(approximate)}\ \underset{t\in\mathbb{R}^n}{\arg\min}\ (g_k + B_k n_k)^T t + \tfrac{1}{2} t^T B_k t \quad \text{subject to}\ A_k t = 0 \ \text{and}\ \|n_k + t\| \le \Delta_k$$

Snags:

- choice of σ_k
- incompatible constraints

CONSTRAINT REDUCTION — BYRD–OMOJOKUN

normal step: replace

$$A_k s = -c_k \text{ and } \|s\| \leq \Delta_k$$

by

approximately minimize $\|A_k n + c_k\|$ subject to $\|n\| \leq \Delta_k$

tangential step: as in Vardi

- use conjugate gradients to solve both subproblems
  $\Longrightarrow$ Cauchy points in both cases
- globally convergent using ℓ_2 merit function
- basis of successful `KNITRO` package

CONSTRAINT LUMPING — CELIS–DENNIS–TAPIA

normal step: replace

$$A_k s = -c_k \text{ and } \|s\| \leq \Delta_k$$

by

$$\|A_k n + c_k\| \leq \sigma_k \text{ and } \|n\| \leq \Delta_k$$

where $\sigma_k \in [0, \|c_k\|]$ is large enough so that there is a feasible n_k

tangential step:

$$\text{(approximate)} \arg\min_{t \in \mathbb{R}^n} \; (g_k + B_k n_k)^T t + \tfrac{1}{2} t^T B_k t$$
$$\text{subject to} \quad \|A_k t + A_k n_k + c_k\| \leq \sigma_k \text{ and } \|t + n_k\| \leq \Delta_k$$

Snags:

- choice of σ_k
- tangential subproblem is (NP?) hard

FILTER METHODS — FLETCHER AND LEYFFER

Rationale:

- trust-region and linearized constraints compatible if c_k is small enough so long as $c(x) = 0$ is compatible
  $\Longrightarrow$ if trust-region subproblem incompatible, simply move closer to constraints
- merit functions depend on arbitrary parameters
  $\Longrightarrow$ use a different mechanism to measure progress

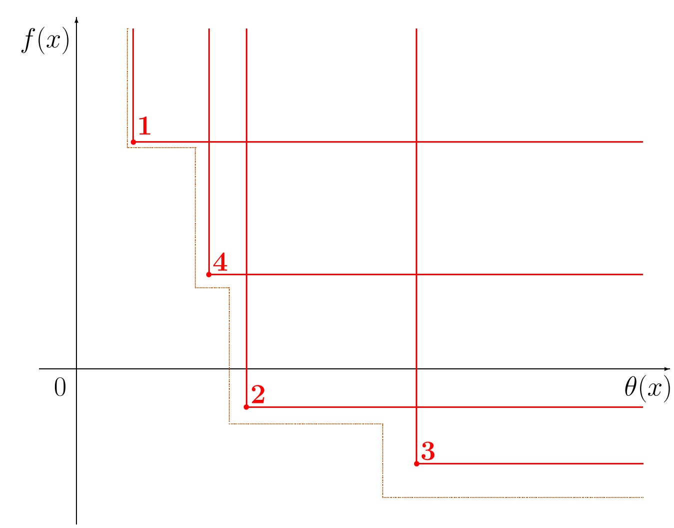
Let $\theta = \|c(x)\|$

A **filter** is a set of pairs $\{(\theta_k, f_k)\}$ such that no member dominates another, i.e., it does not happen that

$$\theta_i \text{“}<\text{”} \theta_j \text{ and } f_i \text{“}<\text{”} f_j$$

for any pair of filter points $i \neq j$

A FILTER WITH FOUR ENTRIES



BASIC FILTER METHOD

- if possible find

  $$s_k = \arg\min_{s \in \mathbb{R}^n} g_k^T s + \tfrac{1}{2} s^T B_k s \ \text{ subject to } \ A_k s = -c_k \ \text{ and } \ \|s\| \leq \Delta_k$$

  otherwise, find s_k:

  $$\theta(x_k + s_k) \text{“}<\text{”} \theta_i \ \text{ for all } \ i \leq k$$

- if $x_k + s_k$ is “acceptable” for the filter, set $x_{k+1} = x_k + s_k$ and possibly increase Δ_k and “prune” filter
- otherwise reduce Δ_k and try again

In practice, far more complicated than this!