Equivalent saddle-point problems

Working note RAL-NA-2007–1 — Nicholas I. M. Gould

14th June 2007

1 Equivalent problems

Consider symmetric saddle-point problems of the form

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{1.1}$$

Then since $Ax + B^T y = a$ and $Bx - Cy = b$, the solution to (1.1) also satisfies the symmetric system[1]

$$\begin{aligned} \left[\sigma \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} + \begin{pmatrix} A \\ B \end{pmatrix} D (A \;\; B^T) + \begin{pmatrix} B^T \\ -C \end{pmatrix} E (B \;\; -C) \right] \begin{pmatrix} x \\ y \end{pmatrix} \\ = \sigma \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} A \\ B \end{pmatrix} Da + \begin{pmatrix} B^T \\ -C \end{pmatrix} Eb \end{aligned} \tag{1.2}$$

for given real σ and arbitrary symmetric matrices D and E. We denote the coefficient matrix of (1.2) as $K(\sigma, D, E)$.

We observe that this general form allows us to reproduce many existing alternatives to (1.1). In particular

- $\sigma = -1$, $D = A^{-1}$ and $E = 0$ gives the Schur-complement method for finding y. Note that $K(-1, A^{-1}, 0)$ is singular.
- $\sigma = -1$, $D = A_0^{-1}$ and $E = 0$ for given A_0 gives the method of Bramble and Pasciak [1].
- $\sigma = \gamma$, $D = I$, $E = -I$ for given γ gives Liesens' method [3].
- $\sigma = -(\alpha + \beta\gamma)$, $D = \alpha A_0^{-1} + \beta I$ and $E = -\beta I$ gives the combination method of Stoll and Wathen [4].
- $\sigma = 1$, $D = 0$ and $E = (1 + \nu) C^{-1}$ for given ν (and in particular $\nu = 1$) is the method proposed by Forsgren, Gill and Griffin [2].

[1]Strictly σ can be absorbed into D and E, but for compatibility with existing results we choose not to do so.

- $\sigma = 0$, $D = I$, $E = I$ gives the method of normal equations.
- $\sigma = 0$, $D = A^{-1}$, $E = C^{-1}$ gives the primal-dual Schur-complement method for simultaneously finding x and y.

This general form (1.2) would then seem the "natural" framework in which to study alternatives to (1.1).

Questions:

- can one choose σ, D and E so that $K(\sigma, D, E)$ is positive definite? In particular what if A and C are singular? What if A is indefinite but $K(1, 0, 0)$ has the "right inertia"?
- can one analyse the spectrum of $K(\sigma, D, E)$?
- if the spectrum is poor, can one precondition?

2 Extensions

Much the same can be done for the non-symmetric block system

$$\begin{pmatrix} A & F \\ B & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.1}$$

Then since $Ax + Fy = a$ and $Bx - Cy = b$, the solution to (2.1) also satisfies the block system

$$\begin{aligned} \left[\sigma \begin{pmatrix} A & F \\ B & -C \end{pmatrix} + \begin{pmatrix} G \\ H \end{pmatrix} (A \;\; B^T) + \begin{pmatrix} M \\ N \end{pmatrix} (B \;\; -C) \right] \begin{pmatrix} x \\ y \end{pmatrix} \\ = \sigma \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} G \\ H \end{pmatrix} a + \begin{pmatrix} M \\ N \end{pmatrix} b \end{aligned} \tag{2.2}$$

for all $\sigma \neq 0$ and arbitrary matrices G, H, M and N of the correct dimension.

References

[1] J. H. Bramble and J. E. Pasciak. A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. *Mathematics of Computation*, 50:1–17, 1988.

[2] A. Forsgren, P. E. Gill, and J. D. Griffin. Iterative solution of augmented systems arising in interior methods. *SIAM Journal on Optimization*, 18(2):666–690, 2007.

[3] J. Liesen. A note on the eigenvalues of saddle point matrices. Technical Report 10-2006, Institut fuer Mathematik, Technische Universität Berlin, 2006.

[4] M. Stoll and A. J. Wathen. Combination preconditioning and self-adjointness in non-standard inner products with application to saddle point problems. Technical Report NA-07/11, Oxford University Computing Laboratory, Oxford, England, 2007.