

## Notes for Part 6: Interior-point methods for inequality constrained optimization

Nick Gould, CSED, RAL, Chilton, OX11 0QX, England (n.gould@rl.ac.uk)

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### 6 Sketches of proofs for Part 6

#### 6.1 Proof of Theorem 6.1

Let  $\mathcal{A} = \mathcal{A}(x_*)$ , and  $\mathcal{I} = \{1, \dots, m\} \setminus \mathcal{A}$  be the indices of constraints that are active and inactive at  $x_*$ . Furthermore let subscripts  $\mathcal{A}$  and  $\mathcal{I}$  denote the rows of matrices/vectors whose indices are indexed by these sets. Denote the left generalized inverse of  $A_{\mathcal{A}}^T(x)$  by

$$A_{\mathcal{A}}^+(x) = \left( A_{\mathcal{A}}(x) A_{\mathcal{A}}^T(x) \right)^{-1} A_{\mathcal{A}}(x)$$

at any point for which  $A_{\mathcal{A}}(x)$  is full rank. Since, by assumption,  $A_{\mathcal{A}}(x_*)$  is full rank, these generalized inverses exists, and are bounded and continuous in some open neighbourhood of  $x_*$ .

Now let

$$(y_k)_i = \frac{\mu_k}{c_i(x_k)}$$

for  $i = 1, \dots, m$ , as well as

$$(y_*)_{\mathcal{A}} = A_{\mathcal{A}}^+(x_*) g(x_*)$$

and  $(y_*)_{\mathcal{I}} = 0$ . If  $\mathcal{I} \neq \emptyset$ ,

$$\|(y_k)_{\mathcal{I}}\|_2 \leq 2\mu_k \sqrt{|\mathcal{I}|} / \min_{i \in \mathcal{I}} |c_i(x_*)| \quad (6.1)$$

for all sufficiently large  $k$ . It then follows from the inner-iteration termination test that

$$\begin{aligned} \|g(x_k) - A_{\mathcal{A}}^T(x_k)(y_k)_{\mathcal{A}}\|_2 &\leq \|g(x_k) - A^T(x_k)y_k\|_2 + \|A_{\mathcal{I}}^T(x_k)(y_k)_{\mathcal{I}}\|_2 \\ &\leq \bar{\epsilon}_k \stackrel{\text{def}}{=} \epsilon_k + \mu_k \min_{i \in \mathcal{I}} |c_i(x_*)|. \end{aligned} \quad (6.2)$$

Hence

$$\|A_{\mathcal{A}}^+(x_k)g(x_k) - (y_k)_{\mathcal{A}}\|_2 = \|A_{\mathcal{A}}^+(x_k)(g(x_k) - A_{\mathcal{A}}^T(x_k)(y_k)_{\mathcal{A}})\|_2 \leq 2\|A_{\mathcal{A}}^+(x_*)\|_2 \bar{\epsilon}_k.$$

Then

$$\|(y_k)_{\mathcal{A}} - (y_*)_{\mathcal{A}}\|_2 \leq \|A_{\mathcal{A}}^+(x_*)g(x_*) - A_{\mathcal{A}}^+(x_k)g(x_k)\|_2 + \|A_{\mathcal{A}}^+(x_k)g(x_k) - (y_k)_{\mathcal{A}}\|_2$$

which, in combination with (6.1), implies that  $\{y_k\}$  converges to  $y_*$ . In addition, continuity of the gradients and (6.2) implies that

$$g(x_*) - A^T(x_*)y_* = 0$$

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while the fact that  $c(x_k) > 0$  for all  $k$ , the definition of  $y_k$  and  $y_*$  (and the implication that  $c_i(x_k)(y_k)_i = \mu_k$ ) shows that  $c(x_*) \geq 0$ ,  $y_* \geq 0$  and  $c_i(x_*)(y_*)_i = 0$ . Hence  $(x_*, y_*)$  satisfies the first-order optimality conditions.

#### 6.2 Proof of Theorem 6.2

The proof of this result is elementary, but rather long and involved. See

N. Gould, D. Orban, A. Sartenaer and Ph. L. Toint, "Superlinear convergence of primal-dual interior point algorithms for nonlinear programming", *SIAM J. Optimization*, **11**(4) (2001) 974:1002

for full details.

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