Part 6: Interior-point methods
for inequality constrained optimization

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$$\underset{x\in \mathbb{R}^n}{\text{minimize}} \quad f(x) \;\; \text{subject to} \;\; c(x) \geq 0$$

Part C course on continuoue optimization

CONSTRAINED MINIMIZATION

$$\underset{x\in \mathbb{R}^n}{\text{minimize}} \; f(x) \;\; \text{subject to} \;\; c(x) \geq 0$$

where the **objective function** $f : \mathbb{R}^n \longrightarrow \mathbb{R}$
and the **constraints** $c : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

- assume that $f,\; c \in C^1$ (sometimes C^2) and Lipschitz
- often in practice this assumption violated, but not necessary

CONSTRAINTS AND MERIT FUNCTIONS

Two conflicting goals:

- minimize the objective function $f(x)$
- satisfy the constraints

Recall — overcome this by minimizing a composite **merit function** $\Phi(x, p)$ for which

- p are parameters
- (some) minimizers of $\Phi(x, p)$ wrt x approach those of $f(x)$ subject to the constraints as p approaches some set $\mathcal{P}$
- only uses **unconstrained** minimization methods

A MERIT F$^{\underline{\text{n}}}$ FOR INEQUALITY CONSTRAINTS

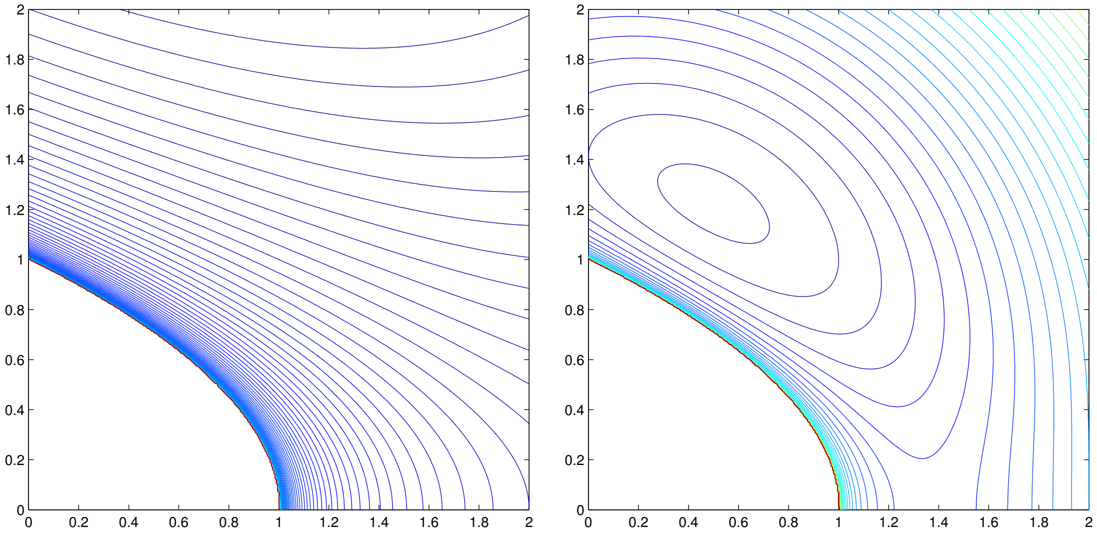
$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; f(x) \;\; \text{subject to} \;\; c(x) \geq 0$$

Merit function **(logarithmic barrier function)**:

$$\Phi(x, \mu) = f(x) - \mu \sum_{i=1}^{m} \log c_i(x)$$

- required solution as μ approaches $\{0\}$ from above
- may have other useless stationary points
- requires a strictly interior point to start
- consequent points are interior

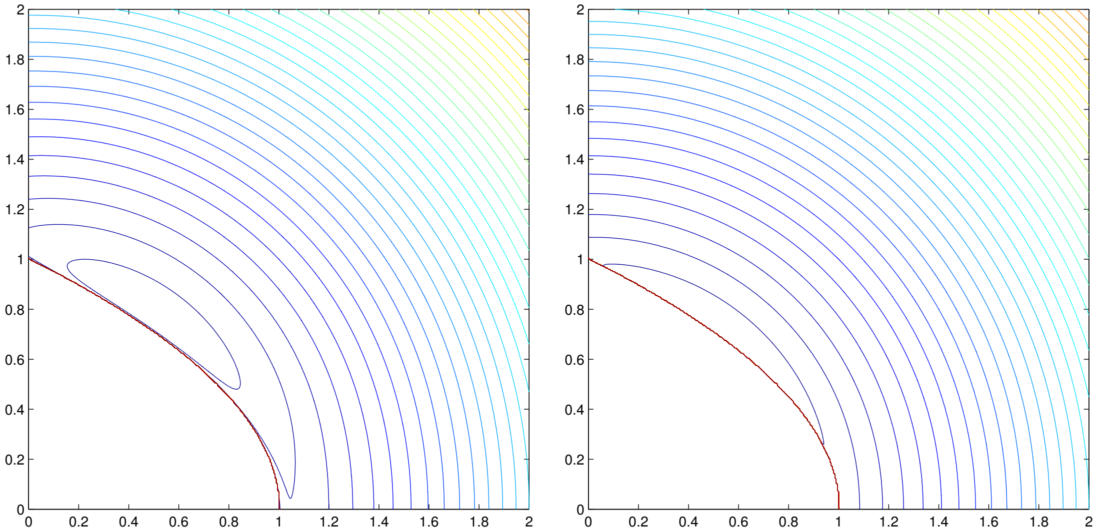
CONTOURS OF THE BARRIER FUNCTION



$\mu = 10$ $\mu = 1$

Barrier function for $\min x_1^2 + x_2^2$ subject to $x_1 + x_2^2 \geq 1$

CONTOURS OF THE BARRIER FUNCTION (cont.)



$\mu = 0.1$ $\mu = 0.01$

Barrier function for $\min x_1^2 + x_2^2$ subject to $x_1 + x_2^2 \geq 1$

BASIC BARRIER FUNCTION ALGORITHM

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Given $\mu_0 > 0$, set $k = 0$
Until "convergence" iterate:
    Find $x^{\rm s}_k$ for which $c(x^{\rm s}_k) > 0$
    Starting from $x^{\rm s}_k$, use an unconstrained
      minimization algorithm to find an
      "approximate" minimizer $x_k$ of $\Phi(x, \mu_k)$
    Compute $\mu_{k+1} > 0$ smaller than $\mu_k$ such
      that $\lim_{k \rightarrow \infty} \mu_{k+1} = 0$ and increase $k$ by 1
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- ⊙ often choose $\mu_{k+1} = 0.1\mu_k$ or even $\mu_{k+1} = \mu_k^2$
- ⊙ might choose $x^{\rm s}_{k+1} = x_k$

MAIN CONVERGENCE RESULT

The **active set** $\mathcal{A}(x) = \{i \mid c_i(x) = 0\}$

Theorem 6.1. Suppose that f, $c \in \mathcal{C}^2$, that $(y_k)_i \stackrel{\rm def}{=} \mu_k / c_i(x_k)$ for $i = 1, \ldots, m$, that

$$\|\nabla_x \Phi(x_k, \mu_k)\|_2 \leq \epsilon_k$$

where ϵ_k converges to zero as $k \rightarrow \infty$, and that x_k converges to x_* for which $\{a_i(x_*)\}_{i \in \mathcal{A}(x_*)}$ are linearly independent. Then x_* satisfies the first-order necessary optimality conditions for the problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; f(x) \;\; \text{subject to} \;\; c(x) \geq 0$$

and $\{y_k\}$ converge to the associated Lagrange multipliers y_*.

PROOF OF THEOREM 6.1

Let $\mathcal{M} \stackrel{\text{def}}{=} \{1, \ldots, m\}$, $\mathcal{A} \stackrel{\text{def}}{=} \{i \mid c_i(x_*) = 0\}$ and $\mathcal{I} \stackrel{\text{def}}{=} \mathcal{M} \setminus \mathcal{A}$.
Generalized inv. $A_{\mathcal{A}}^+(x) \stackrel{\text{def}}{=} \left(A_{\mathcal{A}}(x) A_{\mathcal{A}}^T(x)\right)^{-1} A_{\mathcal{A}}(x)$ bounded near x_*.
Define

$$(y_k)_i = \frac{\mu_k}{c_i(x_k)}, i \in \mathcal{M}, \quad (y_*)_{\mathcal{A}} = A_{\mathcal{A}}^+(x_*) g(x_*) \text{ and } (y_*)_{\mathcal{I}} = 0.$$

$$\|(y_k)_{\mathcal{I}}\|_2 \leq 2\mu_k \sqrt{|\mathcal{I}|} / \min_{i \in \mathcal{I}} |c_i(x_*)| \tag{1}$$

(if $\mathcal{I} \neq \emptyset$) for all sufficiently large k. (1) + inner-it. termination $\Longrightarrow$

$$\begin{aligned} \|g(x_k) - A_{\mathcal{A}}^T(x_k)(y_k)_{\mathcal{A}}\|_2 \leq & \ \|g(x_k) - A^T(x_k) y_k\|_2 + \|A_{\mathcal{I}}^T(x_k)(y_k)_{\mathcal{I}}\|_2 \\ \leq & \ \bar{\epsilon}_k \stackrel{\text{def}}{=} \epsilon_k + \mu_k \frac{2\sqrt{|\mathcal{I}|}\|A_{\mathcal{I}}\|_2}{\min_{i \in \mathcal{I}} |c_i(x_*)|} \end{aligned} \tag{2}$$

$$\begin{aligned} \Longrightarrow \|A_{\mathcal{A}}^+(x_k) g(x_k) - (y_k)_{\mathcal{A}}\|_2 = & \ \|A_{\mathcal{A}}^+(x_k)(g(x_k) - A_{\mathcal{A}}^T(x_k)(y_k)_{\mathcal{A}})\|_2 \\ \leq & \ 2\|A_{\mathcal{A}}^+(x_*)\|_2 \bar{\epsilon}_k \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad & \|(y_k)_{\mathcal{A}} - (y_*)_{\mathcal{A}}\|_2 \\ & \leq \|A_{\mathcal{A}}^+(x_*) g(x_*) - A_{\mathcal{A}}^+(x_k) g(x_k)\|_2 + \|A_{\mathcal{A}}^+(x_k) g(x_k) - (y_k)_{\mathcal{A}}\|_2 \end{aligned}$$

+ (1) $\Longrightarrow$ $\{y_k\} \longrightarrow y_*$. Continuity of gradients + (2) $\Longrightarrow$

$$g(x_*) - A^T(x_*) y_* = 0$$

$c(x_k) > 0$, defs. of y_k and y_* + $c_i(x_k)(y_k)_i = \mu_k \Longrightarrow$
$c(x_*) \geq 0$, $y_* \geq 0$ and $c_i(x_*)(y_*)_i = 0$.
$\Longrightarrow$ (x_*, y_*) satisfies the first-order optimality conditions.

ALGORITHMS TO MINIMIZE $\Phi(x, \mu)$

Can use

- linesearch methods
  - should use specialized linesearch to cope with singularity of log
- trust-region methods
  - need to reject points for which $c(x_k + s_k) \not> 0$
  - (ideally) need to "shape" trust region to cope with contours of the singularity

DERIVATIVES OF THE BARRIER FUNCTION

- $\nabla_x \Phi(x, \mu) = g(x, y(x))$
- $$\begin{aligned}\nabla_{xx} \Phi(x, \mu) &= H(x, y(x)) + \mu A^T(x) C^{-2}(x) A(x) \\ &= H(x, y) + A^T(x) C^{-1}(x) Y(x) A(x) \\ &= H(x, y) + \tfrac{1}{\mu} A^T(x) Y^2(x) A(x)\end{aligned}$$

where

- **Lagrange multiplier estimates**: $y(x) = \mu C^{-1}(x) e$ where e is the vector of ones
- $C(x) = \text{diag}(c_1(x), \ldots, c_m(x))$
- $Y(x) = \text{diag}(y_1(x), \ldots, y_m(x))$
- $g(x, y(x)) = g(x) - A^T(x) y(x)$: **gradient of the Lagrangian**
- $H(x, y(x)) = H(x) - \sum_{i=1}^{m} y_i(x) H_i(x)$: **Lagrangian Hessian**

LIMITING DERIVATIVES OF Φ

Let $\mathcal{I}$ = inactive set at $x_* = \{1, \ldots, m\} \setminus \mathcal{A}$
For small μ: roughly

$$\begin{aligned}\nabla_x \Phi(x,\mu) &= \underbrace{g(x) - A_{\mathcal{A}}^T(x) Y_{\mathcal{A}}^{-1}(x) e}_{\text{moderate}} - \underbrace{\mu A_{\mathcal{I}}^T(x) C_{\mathcal{I}}^{-1}(x) e}_{\text{small}} \\ &\approx g(x) - A_{\mathcal{A}}^T(x) Y_{\mathcal{A}}^{-1}(x) e\end{aligned}$$

$$\begin{aligned}\nabla_{xx} \Phi(x,\mu) &= \underbrace{H(x,y(x))}_{\text{moderate}} + \underbrace{\mu A_{\mathcal{I}}^T(x) C_{\mathcal{I}}^{-2}(x) A_{\mathcal{I}}(x)}_{\text{small}} + \underbrace{\frac{1}{\mu} A_{\mathcal{A}}^T(x) Y_{\mathcal{A}}^2(x) A_{\mathcal{A}}(x)}_{\text{large}} \\ &\approx \frac{1}{\mu} A_{\mathcal{A}}^T(x) Y_{\mathcal{A}}^2(x) A_{\mathcal{A}}(x) \\ &= A_{\mathcal{A}}^T(x) C_{\mathcal{A}}^{-1}(x) Y_{\mathcal{A}}(x) A_{\mathcal{A}}(x) \\ &= \mu A_{\mathcal{A}}^T(x) C_{\mathcal{A}}^{-2}(x) A_{\mathcal{A}}(x)\end{aligned}$$

GENERIC BARRIER NEWTON SYSTEM

Newton correction s from x for barrier function is

$$\big(H(x,y(x)) + A^T(x) C^{-1}(x) Y(x) A(x)\big)\, s = -g(x,y(x))$$

LIMITING NEWTON METHOD

For small μ: roughly

$$\mu A_{\mathcal{A}}^T(x) C_{\mathcal{A}}^{-2}(x) A_{\mathcal{A}}(x) s \approx -\big(g(x) - A_{\mathcal{A}}^T(x) Y_{\mathcal{A}}^{-1}(x) e\big)$$

POTENTIAL DIFFICULTIES I

Ill-conditioning of the Hessian of the barrier function:

roughly speaking (non-degenerate case)

- $\odot$ m_a eigenvalues $\approx \lambda_i \left[A_{\mathcal{A}}^T Y_{\mathcal{A}}^2 A_{\mathcal{A}} \right] / \mu_k$
- $\odot$ $n - m_a$ eigenvalues $\approx \lambda_i \left[N_{\mathcal{A}}^T H(x_*, y_*) N_{\mathcal{A}} \right]$

where

m_a = number of active constraints
$\mathcal{A}$ = active set at x_*
Y = diagonal matrix of Lagrange multipliers
$N_{\mathcal{A}}$ = orthogonal basis for null-space of $A_{\mathcal{A}}$

$\Longrightarrow$ condition number of $\nabla_{xx}\Phi(x_k, \mu_k) = O(1/\mu_k)$
$\Longrightarrow$ may not be able to find minimizer easily

POTENTIAL DIFFICULTIES II

Value $x_{k+1}^{\mathrm{s}} = x_k$ is a poor starting point: Suppose

$$\begin{aligned} 0 \approx & \ \nabla_x \Phi(x_k, \mu_k) = g(x_k) - \mu_k A^T(x_k) C^{-1}(x_k) e \\ \approx & \ g(x_k) - \mu_k A_{\mathcal{A}}^T(x_k) C_{\mathcal{A}}^{-1}(x_k) e \end{aligned}$$

Roughly speaking (non-degenerate case) Newton correction satisfies

$$\mu_{k+1} A_{\mathcal{A}}^T(x_k) C_{\mathcal{A}}^{-2}(x_k) A_{\mathcal{A}}(x_k) s \approx (\mu_{k+1} - \mu_k) A_{\mathcal{A}}^T(x_k) C_{\mathcal{A}}^{-1}(x_k) e$$

$\Longrightarrow$ (full rank)

$$A_{\mathcal{A}}(x_k) s \approx \left(1 - \frac{\mu_k}{\mu_{k+1}} \right) c_{\mathcal{A}}(x_k)$$

$\Longrightarrow$ (Taylor expansion)

$$c_{\mathcal{A}}(x_k + s) \approx c_{\mathcal{A}}(x_k) + A_{\mathcal{A}}(x_k) s \approx \left(2 - \frac{\mu_k}{\mu_{k+1}} \right) c_{\mathcal{A}}(x_k) < 0$$

if $\mu_{k+1} < \frac{1}{2}\mu_k \Longrightarrow$ Newton step infeasible $\Longrightarrow$ slow convergence

PERTURBED OPTIMALITY CONDITIONS

First order optimality conditions for

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; f(x) \;\; \text{subject to} \;\; c(x) \geq 0$$

are:

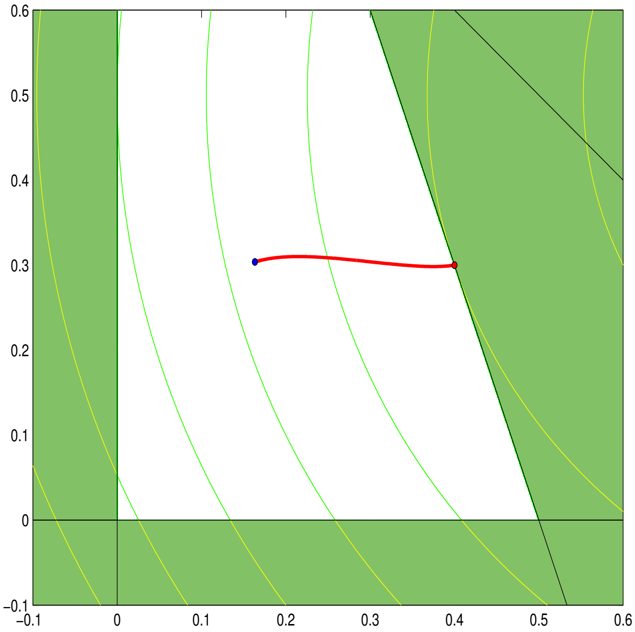
$$\begin{array}{cc} g(x) - A^T(x)y = 0 & \text{dual feasibility} \\ C(x)y = 0 & \text{complementary slackness} \\ c(x) \geq 0 \;\text{ and }\; y \geq 0 & \end{array}$$

Consider the "perturbed" problem

$$\begin{array}{cc} g(x) - A^T(x)y = 0 & \text{dual feasibility} \\ C(x)y = \mu e & \textbf{perturbed}\text{ comp. slkns.} \\ c(x) > 0 \;\text{ and }\; y > 0 & \end{array}$$

where $\mu > 0$

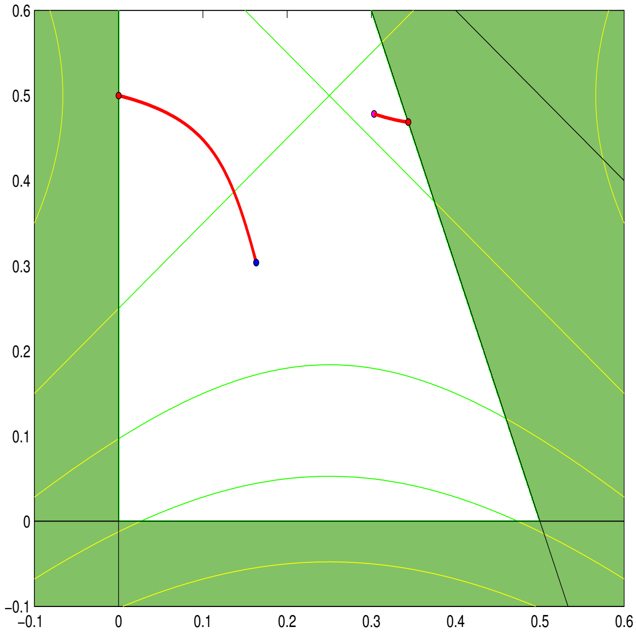
CENTRAL PATH TRAJECTORY



$$\begin{array}{c} \min (x_1 - 1)^2 + (x_2 - 0.5)^2 \\ \text{subject to} \;\; x_1 + x_2 \leq 1 \\ 3x_1 + x_2 \leq 1.5 \\ (x_1, x_2) \geq 0 \end{array}$$

Trajectory $x(\mu)$ of perturbed optimality conditions as μ ranges from infinity down to zero

TRAJECTORIES FOR THE NON-CONVEX CASE



$$\begin{gathered}\min -2(x_1-0.25)^2+2(x_2-0.5)^2\\ \text{subject to } \ x_1+x_2\le 1\\ 3x_1+x_2\le 1.5\\ (x_1,x_2)\ge 0\end{gathered}$$

Trajectories $x(\mu)$ of perturbed optimality conditions
as μ ranges from infinity down to zero

PRIMAL-DUAL PATH-FOLLOWING METHODS

Track roots of

$$g(x)-A^T(x)y=0 \ \text{ and } \ C(x)y-\mu e=0$$

as $0<\mu\longrightarrow 0$, while maintaining $c(x)>0$ and $y>0$

- ⊙ nonlinear system $\Longrightarrow$ use Newton's method

Newton correction (s,w) to (x,y) satisfies

$$\begin{pmatrix} H(x,y) & -A^T(x)\\ YA(x) & C(x)\end{pmatrix}\begin{pmatrix} s\\ w\end{pmatrix}=-\begin{pmatrix} g(x)-A^T(x)y\\ C(x)y-\mu e\end{pmatrix}$$

Eliminate $w\Longrightarrow$

$$\left(H(x,y)+A^T(x)C^{-1}(x)YA(x)\right)s=-\left(g(x)-\mu A^T(x)C^{-1}(x)e\right)$$

c.f. Newton method for barrier minimization!

PRIMAL VS. PRIMAL-DUAL

Primal:

$$\left(H(x,y(x)) + A^T(x)C^{-1}(x)Y(x)A(x)\right) s^{\text{P}} = -g(x,y(x))$$

Primal-dual:

$$\left(H(x,y) + A^T(x)C^{-1}(x)YA(x)\right) s^{\text{PD}} = -g(x,y(x))$$

where

$$y(x) = \mu C^{-1}(x)e$$

What is the difference?

- ⊙ freedom to choose y in $H(x,y) + A^T(x)C^{-1}(x)YA(x)$ for primal-dual … vital
- ⊙ Hessian approximation for small μ
  $$H(x,y) + A^T(x)C^{-1}(x)YA(x) \approx A_{\mathcal{A}}^T(x)C_{\mathcal{A}}^{-1}(x)Y_{\mathcal{A}}A_{\mathcal{A}}(x)$$

POTENTIAL DIFFICULTY II … REVISITED

Value $x_{k+1}^{\text{S}} = x_k$ can be a good starting point:

- ⊙ primal method has to choose $y = y(x_k^{\text{S}}) = \mu_{k+1}C^{-1}(x_k)e$
  - ⋄ factor μ_{k+1}/μ_k too small for a good Lagrange multiplier estimate
- ⊙ primal-dual method can choose $y = \mu_k C^{-1}(x_k)e \longrightarrow y_*$

Advantage: roughly (non-degenerate case) correction s^{PD} satisfies

$$\mu_k A_{\mathcal{A}}^T(x_k)C_{\mathcal{A}}^{-2}(x_k)A_{\mathcal{A}}(x_k)s^{\text{PD}} \approx (\mu_{k+1} - \mu_k)A_{\mathcal{A}}^T(x_k)C_{\mathcal{A}}^{-1}(x_k)e$$

$\Longrightarrow$ (full rank)

$$A_{\mathcal{A}}(x_k)s^{\text{PD}} \approx \left(\frac{\mu_{k+1}}{\mu_k} - 1\right) c_{\mathcal{A}}(x_k)$$

$\Longrightarrow$ (Taylor expansion)

$$c_{\mathcal{A}}(x_k + s^{\text{PD}}) \approx c_{\mathcal{A}}(x_k) + A_{\mathcal{A}}(x_k)s^{\text{PD}} \approx \frac{\mu_{k+1}}{\mu_k} c_{\mathcal{A}}(x_k) > 0$$

$\Longrightarrow$ Newton step allowed $\Longrightarrow$ fast convergence

PRIMAL-DUAL BARRIER METHODS

Choose a search direction s for $\Phi(x, \mu_k)$ by
(approximately) solving the problem

$$\underset{s \in \mathbb{R}^n}{\text{minimize}} \;\; g(x, y(x))^T s + \tfrac{1}{2} s^T \left(H(x, y) + A^T(x) C^{-1}(x) Y A(x) \right) s$$

possibly subject to a trust-region constraint

- ⊙ $y(x) = \mu C^{-1}(x) e \Longrightarrow g(x, y(x)) = \nabla_x \Phi(x, \mu)$
- ⊙ $y = \ldots$
  - ⋄ $y(x) \Longrightarrow$ primal Newton method
  - ⋄ occasionally $(\mu_{k-1}/\mu_k) y(x) \Longrightarrow$ good starting point
  - ⋄ $y^{\text{OLD}} + w^{\text{OLD}} \Longrightarrow$ primal-dual Newton method
  - ⋄ $\max(y^{\text{OLD}} + w^{\text{OLD}}, \epsilon(\mu_k) e)$ for "small" $\epsilon(\mu_k) > 0$
    (e.g., $\epsilon(\mu_k) = \mu_k^{1.5}$) $\Longrightarrow$ practical primal-dual method

POTENTIAL DIFFICULTY I . . . REVISITED

Ill-conditioning $\not\Longrightarrow$ we can't solve equations accurately:
roughly (non-degenerate case, $\mathcal{I}$ = inactive set at x_*)

$$\begin{pmatrix} H & -A^T \\ YA & C \end{pmatrix} \begin{pmatrix} s \\ w \end{pmatrix} = - \begin{pmatrix} g - A^T y \\ Cy - \mu e \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} H & -A_{\mathcal{A}}^T & -A_{\mathcal{I}}^T \\ Y_{\mathcal{A}} A_{\mathcal{A}} & C_{\mathcal{A}} & 0 \\ Y_{\mathcal{I}} A_{\mathcal{I}} & 0 & C_{\mathcal{I}} \end{pmatrix} \begin{pmatrix} s \\ w_{\mathcal{A}} \\ w_{\mathcal{I}} \end{pmatrix} = - \begin{pmatrix} g - A_{\mathcal{A}}^T y_{\mathcal{A}} - A_{\mathcal{I}}^T y_{\mathcal{I}} \\ C_{\mathcal{A}} y_{\mathcal{A}} - \mu e \\ C_{\mathcal{I}} y_{\mathcal{I}} - \mu e \end{pmatrix} \Longrightarrow$$

$$\begin{pmatrix} H + A_{\mathcal{I}}^T C_{\mathcal{I}}^{-1} Y_{\mathcal{I}} A_{\mathcal{I}} & -A_{\mathcal{A}}^T \\ A_{\mathcal{A}} & C_{\mathcal{A}} Y_{\mathcal{A}}^{-1} \end{pmatrix} \begin{pmatrix} s \\ w_{\mathcal{A}} \end{pmatrix} = - \begin{pmatrix} g - A_{\mathcal{A}}^T y_{\mathcal{A}} - \mu A_{\mathcal{I}}^T C_{\mathcal{I}}^{-1} e \\ c_{\mathcal{A}} - \mu Y_{\mathcal{A}}^{-1} e \end{pmatrix}$$

- ⊙ potentially bad terms $C_{\mathcal{I}}^{-1}$ and $Y_{\mathcal{A}}^{-1}$ bounded
- ⊙ in the limit becomes well-behaved

$$\begin{pmatrix} H & -A_{\mathcal{A}}^T \\ A_{\mathcal{A}} & 0 \end{pmatrix} \begin{pmatrix} s \\ w_{\mathcal{A}} \end{pmatrix} = - \begin{pmatrix} g - A_{\mathcal{A}}^T y_{\mathcal{A}} \\ 0 \end{pmatrix}$$

PRACTICAL PRIMAL-DUAL METHOD

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Given $\mu_0 > 0$ and feasible $(x_0^{\mathrm{s}}, y_0^{\mathrm{s}})$, set $k = 0$
Until "convergence" iterate:
  Inner minimization: starting from $(x_k^{\mathrm{s}}, y_k^{\mathrm{s}})$, use an
    unconstrained minimization algorithm to find $(x_k, y_k)$ for which
      $\|C(x_k)y_k - \mu_k e\| \leq \mu_k$ and $\|g(x_k) - A^T(x_k)y_k\| \leq \mu_k^{1.00005}$
  Set $\mu_{k+1} = \min(0.1\mu_k, \mu_k^{1.9999})$
  Find $(x_{k+1}^{\mathrm{s}}, y_{k+1}^{\mathrm{s}})$ using a primal-dual Newton step from $(x_k, y_k)$
  If $(x_{k+1}^{\mathrm{s}}, y_{k+1}^{\mathrm{s}})$ is infeasible, reset $(x_{k+1}^{\mathrm{s}}, y_{k+1}^{\mathrm{s}})$ to $(x_k, y_k)$
  Increase $k$ by 1
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FAST ASYMPTOTIC CONVERGENCE

Theorem 6.2. Suppose that f, $c \in \mathcal{C}^2$, that a subsequence $\{(x_k, y_k)\}$, $k \in \mathcal{K}$, of the practical primal-dual method converges to (x_*, y_*) satisfying second-order sufficiency conditions, that $A_{\mathcal{A}}(x_*)$ is full-rank, and that $(y_*)_{\mathcal{A}} > 0$. Then the starting point satisfies the inner-minimization termination test (i.e., $(x_k, y_k) = (x_k^{\mathrm{s}}, y_k^{\mathrm{s}})$) and the whole sequence $\{(x_k, y_k)\}$ converges to (x_*, y_*) at a superlinear rate (Q-factor 1.9998).

OTHER ISSUES

- polynomial algorithms for many convex problems
    - linear programming
    - quadratic programming
    - semi-definite programming . . .
- excellent practical performance
- globally, need to keep away from constraint boundary until near convergence, otherwise very slow
- initial interior point:

$$\underset{(x,c)}{\text{minimize}} \; e^T c \;\; \text{subject to} \;\; c(x) + c \geq 0$$