The Fundamental Theorem of Linear Inequalities

Lecture 8, Continuous Optimisation
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Notes by Dr Raphael Hauser (hauser@comlab.ox.ac.uk)

In unconstrained optimisation we found that we can use the optimality conditions derived in Lecture 1 to transform optimisation problems into zero-finding problems for systems of nonlinear equations

$$\nabla f(x) = 0.$$

We will spend the next few lectures to develop a similar approach to constrained optimisation: in this case the optimal solutions can be characterised by systems of nonlinear equations and inequalities.

Constrained Optimisation and the Need for Optimality Conditions:

In the remaining part of this course we will consider the problem of minimising objective functions over constrained domains. The general problem of this kind can be written in the form

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f(x) \\ \text{s.t.} \quad & g_i(x) = 0 \quad (i \in \mathcal{E}), \\ & g_j(x) \geq 0 \quad (i \in \mathcal{I}), \end{aligned}$$

where $\mathcal{E}$ and $\mathcal{I}$ are the finite index sets corresponding to the equality and inequality constraints, and where $f, g_i \in C^k(\mathbb{R}^n, \mathbb{R})$ for all $(i \in \mathcal{I} \cup \mathcal{E})$.

A natural by-product of this analysis will be the notion of a *Lagrangian dual* of an optimisation problem: every optimisation problem - called the primal - has a sister problem in the space of Lagrange multipliers - called the dual.

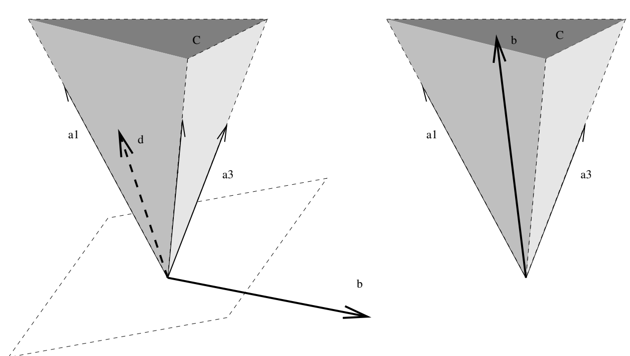
In constrained optimisation it is often advantageous to think of the primal and dual in a combined primal-dual framework where each sheds light from a different angle on a certain saddle-point finding problem.

First we will take a closer look at systems of *linear* inequalities and prove a theorem that will be of fundamental importance in everything that follows:

Theorem 1: Fundamental theorem of linear inequalities. Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$ be a set of vectors. Then exactly one of the two following alternatives occurs:

(I) $\exists y \in \mathbb{R}^m_+$ such that $b = \sum_i^m y_i a_i$.

(II) $\exists d \in \mathbb{R}^n$ such that $d^\top b < 0$ and $d^\top a_i \geq 0$ for all $(i = 1, \ldots, m)$.



Note that Alternative (I) says that b lies in the convex cone generated by the vectors a_i:

$$b \in \text{cone}(a_1, \ldots, a_m) := \left\{ \sum_{i=1}^{n} \lambda_i a_i : \lambda_i \geq 0 \; \forall i \right\}.$$

Alternative (II) on the other hand says that the hyperplane $d^\perp := \{x \in \mathbb{R}^n : d^\top x = 0\}$ strictly separates b from the convex set $\text{cone}(a_1, \ldots, a_m)$.

Thus, Theorem 1 is a result about convex separation: either b is a member of $\text{cone}(a_1, \ldots, a_m)$ or there exists a hyperplane that strictly separates the two objects.

Lemma 1: The two alternatives of Theorem 1 are mutually exclusive.

Proof: If this is not the case then we find the contradiction

$$0 \leq \sum_{i=1}^{m} y_i (d^\top a_i) = d^\top \left(\sum_{i=1}^{m} y_i a_i \right) = d^\top b < 0. \quad \square$$

Lemma 2: W.l.o.g. we may assume that span$\{a_1, \dots, a_m\} = \mathbb{R}^n$.

Proof:

- If span$\{a_1, \dots, a_m\} \neq \mathbb{R}^n$ then either $b \in$ span$\{a_1, \dots, a_m\}$ and then we can restrict all arguments of the proof of Theorem 1 to the linear subspace span$\{a_1, \dots, a_m\}$ of $\mathbb{R}^n$.

- Else, if $b \notin$ span$\{a_1, \dots, a_m\}$ then b cannot be written in the form $b = \sum_i^m \mu_i a_i$, so Alternative (I) does not hold.

Because of Lemma 2, we will henceforth assume that

$$\text{span}\{a_1, \dots, a_m\} = \mathbb{R}^n.$$

We will next construct an algorithm that stops when a situation corresponding to either Alternative (I) or (II) is detected.

This will in fact be the simplex algorithm for LP in disguised form.

- It remains to show that Alternative (II) applies in this case. Let π be the the orthogonal projection of $\mathbb{R}^n$ onto span$\{a_1, \dots, a_{m}$ and let $d = \pi(b) - b$. Then $d \perp$ span$\{a_1, \dots, a_m\}$, so that

$$d^\top b = d^\top(b - \pi(b)) + d^\top \pi(b) = -\|d\|^2 + 0 < 0,$$
$$d^\top a_i = 0 \quad \forall i.$$

Therefore, Alternative (II) holds. □

Algorithm 1:

S0 Choose $J^1 \subseteq \{1, \dots, m\}$ such that span$\{a_i\}_{J^1} = \mathbb{R}^n$, $|J^1| = n$.

S1 For $k = 1, 2, \dots$ repeat

1. decompose $b = \sum_{i \in J^k} y_i^k a_i$
2. if $y_i^k \geq 0 \, \forall i \in J^k$ return y^k and stop.
3. else begin

   let $j^k :=$ min$\{i \in J^k : \, y_i^k < 0\}$

   let $\pi^k : \mathbb{R}^n \to$ span$\{a_i : \, i \in J^k \setminus \{j^k\}\}$ orthogonal projection let $d^k := \|a_{j^k} - \pi^k(a_{j^k})\|^{-1}(a_{j^k} - \pi^k(a_{j^k}))$

if $(d^k)^\top a_i \geq 0$ for $(i = 1, \ldots, m)$ return d^k and stop.

end

4. let $l^k := \min\{i : (d^k)^\top a_i < 0\}$

5. let $J^{k+1} := J^k \setminus \{j^k\} \cup \{l^k\}$

end.

Comments:

- If the algorithm returns y^k in Step 2, then Alternative (I) holds: let $y_i = 0$ for $i \neq J^k$ and $y_i = y_i^k$ for $i \in J$. Then $y \in \mathbb{R}_+^m$ and $b = \sum_i y_i a_i$.

- If the algorithm enters Step 3, then $\{i \in J^k : y_i^k < 0\} \neq \emptyset$ because the condition of Step 2 is not satisfied.

- The vector d^k constructed in Step 3 satisfies
$$(d^k)^\top b = \sum_{i \in J^k} y_i^k (d^k)^\top a_i = y_{j^k}^k (d^k)^\top a_{j^k} < 0, \tag{1}$$
Therefore, if the algorithm returns d^k then Alternative (II) holds with $d = d^k$.

- If the algorithm enters Step 4 then $\{i : (d^k)^\top a_i < 0\} \neq \emptyset$ because the condition of the last "if" statement of Step 3 is not satisfied. Moreover, since
$$(d^k)^\top a_{j^k} = 1,$$
$$(d^k)^\top a_i = 0 \quad (i \in J^k \setminus \{j^k\}),$$
we have $\{i : (d^k)^\top a_i < 0\} \cap J^k = \emptyset$. This shows that $l^k \notin J^k$.

- We have $\text{span}\{a_i : i \in J^{k+1}\} = \mathbb{R}^n$, because $(d^k)^\top a_{l^k} \neq 0$ and $(d^k)^\top a_i = 0$ $(i \in J^k \setminus \{j^k\})$ show that $a_{l^k} \notin \text{span}\{a_i : i \in J^k \setminus \{j^k\}\}$. Moreover, $|J^{k+1}| = n$.

Lemma 3: It can never occur that $J^k = J^t$ for $k < t$.

Proof:

- Let us assume to the contrary that $J^k = J^t$ for some iterations $k < t$, and let $j^{\max} := \max\{j^s : k \leq s \leq t-1\}$.

- Then there exists $p \in \{k, k+1, \ldots, t-1\}$ such that $j^{\max} = j^p$.

- Since $J^k = J^t$, there also exists $q \in \{k, k+1, \ldots, t-1\}$ such that $j^{\max} = l^q$. In other words, the index must have once left J and then reentered, or else it must have entered and then left again.

- Now $j^{\max} = j^p$ implies that for all $i \in J^p$ such that $i < j^{\max}$ we have $y_i^p \geq 0$.

- Likewise, for the same indices i we have $(d^q)^\top a_i \geq 0$, as
$$i < j^{\max} = l^q = \min\{i : (d^q)^\top a_i < 0\}.$$

- Furthermore, we have $y^p_{j^{\max}} = y^p_{j^p} < 0$ and $(d^q)^\top a_{j^{\max}} = (d^q)^\top a_{l^q} < 0$.

- And finally, since $J^s \cap \{j^{\max} + 1, \dots, m\}$ remains unchanged for $s = k, \dots, t$ we have $(d^q)^\top a_i = 0$ for all $i \in J^p$ such that $i > j^{\max}$.

- Therefore,
$$(d^q)^\top b = \sum_{i \in J^p} y_i^p (d^q)^\top a_i \geq 0. \tag{2}$$

- On the other hand, (1) shows $(d^q)^\top b < 0$, contradicting (2). Thus, indices $k < t$ such that $J^k = J^t$ do not exist. □

We are finally ready to prove the fundamental theorem of linear inequalities:

Proof: (Theorem 1)

- Since $J^k \subseteq \{1, \dots, m\}$ and there are finitely many choices for these index sets and Lemma 3 shows that there are no repetitions in the sequence $J^1, J^2, \dots$, the sequence must be finite.

- But this is only possible if in some iteration k Algorithm 1 either returns y^k, detecting that Alternative (I) holds, or d^k, detecting that Alternative (II) holds. □

The Implicit Function Theorem:

Another fundamental tool we will need is the implicit function theorem.

Before stating the theorem, let us illustrate it with an example:

Example 1: The function $f(x_1, x_2) = x_1^2 + x_2^2 - 1$ has a zero at the point $(1, 0)$ and $\frac{\partial}{\partial x_1} f(1, 0) = 1 \neq 0$. In a neighbourhood of this point the level set $\{(x_1, x_2) : f(x_1, x_2) = 0\}$ can be explicitly parameterised in terms of x_2, that is, there exists a function $h(t)$ such that $f(x_1, x_2) = 0$ if and only if $(x_1, x_2) = (h(t), t)$ for some value of t.

- The level set is nothing else but the unit circle S^1, and for (x_1, x_2) with $x_1 > 0$ we have $f(x_1, x_2) = 0$ if and only if $x_1 = h(x_2)$ where $h(t) = \sqrt{1-t^2}$.

- Thus,
$$S^1 \cap \{x \in \mathbb{R}^2 : x_1 > 0\} = \{(h(t), t) : t \in (-1, 1)\},$$
as claimed.

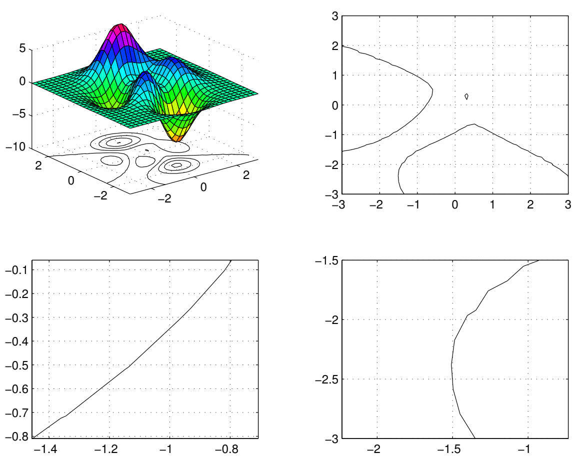
- Another way to say this is that S^1 is a differentiable manifold with local coordinate map
$$\varphi : S^1 \cap \{x \in \mathbb{R}^2 : x_1 > 0\} \to (-1, 1),$$
$$x \mapsto x_2.$$

- The parameterisation in terms of x_2 was only possible because $\frac{\partial}{\partial x_1} f(1, 0) \neq 0$.

- To illustrate this, note that we also have $f(0, 1) = 0$, but now $\frac{\partial}{\partial x_1} f(0, 1) = 0$ and we cannot parameterise S^1 by x_2 in a neighbourhood of $(0, 1)$.

- In fact, in a neighbourhood of $x_2 = 1$, there are two x_1 values $\pm\sqrt{1 - x_2^2}$ such that $f(x_1, x_2) = 0$ when $x_2 < 1$ and none when $x_2 > 0$.

Example 2:



Generalisation:

- $f \in C^k(\mathbb{R}^{p+q}, \mathbb{R}^p)$.

- Let $f'_B(x)$ be the leading $p \times p$ block of the Jacobian matrix $f'(x) = [\, f'_B(x) \; f'_N(x) \,]$,

- and $f'_N(x)$ the trailing $p \times q$ block.

- Let x_B be the first $p \times 1$ block of the vector x

- and x_N the trailing $q \times 1$ block.

Theorem 2: Implicit Function Theorem. Let $f \in C^k(\mathbb{R}^{p+q}, \mathbb{R}^p)$ and let $\bar{x} \in \mathbb{R}^{p+q}$ be such that $f(\bar{x}) = 0$ and $f'_B(\bar{x})$ nonsingular.

Then there exist open neighbourhoods $U_B \subset \mathbb{R}^p$ of $\bar{x}_B$ and $U_N \subset \mathbb{R}^q$ of $\bar{x}_N$ and a function $h \in C^k(U_N, U_B)$ such that for all $(x_B, x_N) \in U_B \times U_N$,

i) $f(x_B, x_N) = 0 \Leftrightarrow x_B = h(x_N)$,

ii) $f'_B(x)$ is nonsingular,

iii) $h'(x_N) = -\left(f'_B(x)\right)^{-1} f'_N(x)$.

Reading Assignment: Lecture-Note 8.