1. (a). Let $\mathbf{x} \in \mathbb{R}^n$. What is a linesearch method for the unconstrained minimization of the objective function $f(\mathbf{x})$? What is the Armijo condition? Briefly describe the backtracking-Armijo linesearch method. [7 marks]

(b). What is the Newton direction? Give two reasons why the Newton direction may not be a good search direction for a linesearch method. [4 marks]

(c). How could the Newton direction be modified to make it suitable for a linesearch method? [4 marks]

(d). Suppose $f(\mathbf{x}) = x_1^2 + 100x_2^2$. What is the Newton direction at $\mathbf{x} = (1, 1)$? Is the Newton direction suitable for a linesearch method? If the initial stepsize for the backtracking-Armijo linesearch is 1, what extra restriction must be imposed on the Armijo condition so that a unit step along the Newton direction is acceptable? [5 marks]

2. (a) Let $\mathbf{x} \in \mathbb{R}^n$. Write down second-order sufficient optimality conditions for a point $\mathbf{x}_*$ to be an isolated local minimizer of $f(\mathbf{x})$ subject to the vector of equality constraints $\mathbf{c}(\mathbf{x}) = \mathbf{0}$. [2 marks]

Consider the equality-constrained problem

$$\underset{\mathbf{s}\in\mathbb{R}^n}{\text{minimize}} \;\; \mathbf{g}^T\mathbf{s} + \tfrac{1}{2}\mathbf{s}^T\mathbf{B}\mathbf{s} \;\; \text{subject to} \;\; \mathbf{A}\mathbf{s} = \mathbf{b},$$

where $n = 3$, $\mathbf{g} = -(1,1,1)^T$, $\mathbf{A} = (0 \;\; 1 \;\; 1)$ and $\mathbf{b} = 2$.

(b) Solve this problem when

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

[Hint, the columns of the matrix

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}$$

are linearly independent and orthogonal to the (single) row of $\mathbf{A}$.] [5 marks]

(c) Solve the problem when

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[4 marks]

(d) What is the SQP correction $\mathbf{s}$ to the primal-dual solution estimate $(\mathbf{x}, \mathbf{y})$ for the problem

$$\underset{\mathbf{x}\in\mathbb{R}^n}{\text{minimize}} \;\; x_1^2 + \tfrac{3}{2}x_3^2 + x_1 + x_2 + x_3 \;\; \text{subject to} \;\; \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_3^2 - 3 = 0$$

when $\mathbf{x} = (0,1,1)$ and $\mathbf{y} = 1$? [5 marks]

(e) What difference would it make if we also imposed the trust-region constraint $\|\mathbf{s}\|_\infty \leq 1/2$ on the SQP step subproblem?
[Hint, recall that $\|\mathbf{s}\|_\infty = \max_{1\leq i\leq n} |s_i|$.] [4 marks]